

ON THE HOMOGENIZATION OF MULTICOMPONENT TRANSPORT

GRÉGOIRE ALLAIRE, HARSHA HUTRIDURGA

ABSTRACT. This paper is devoted to the homogenization of weakly coupled cooperative parabolic systems in strong convection regime with purely periodic coefficients. Our approach is to factor out oscillations from the solution via principal eigenfunctions of an associated spectral problem and to cancel any exponential decay in time of the solution using the principal eigenvalue of the same spectral problem. We employ the notion of two-scale convergence with drift in the asymptotic analysis of the factorized model as the lengthscale of the oscillations tends to zero. This combination of the factorization method and the method of two-scale convergence is applied to upscale an adsorption model for multicomponent flow in an heterogeneous porous medium.

1. INTRODUCTION

Upscaling reactive transport models in porous media is a problem of great practical importance and homogenization theory is a method of choice for achieving this goal (see [15] and references therein). In this paper we focus on a model problem of reactive multicomponent transport for N diluted chemical species in a saturated periodically varying media. The fluid velocity is assumed to be known. On top of usual convective and diffusive effects we consider linear reaction terms which satisfy a specific condition, namely that the reaction matrix is cooperative (see the precise definition in Section 2). This assumption is quite natural for a linear system, as we consider here, since it ensures a maximum (or positivity) principle for solutions which, being concentrations, should indeed be non-negative for obvious physical reasons. As usual the ratio between the period of the coefficients and a characteristic lengthscale of the porous domain is denoted by a small parameter $0 < \varepsilon \ll 1$. Denoting the unknown concentrations by u_α^ε , for $1 \leq \alpha \leq N$, we study in the entire space \mathbb{R}^d the following weakly coupled (i.e., no coupling in the derivatives) system of N parabolic equations with periodic bounded coefficients:

$$(1.1) \quad \rho_\alpha \left(\frac{x}{\varepsilon} \right) \frac{\partial u_\alpha^\varepsilon}{\partial t} + \frac{1}{\varepsilon} b_\alpha \left(\frac{x}{\varepsilon} \right) \cdot \nabla u_\alpha^\varepsilon - \operatorname{div} \left(D_\alpha \left(\frac{x}{\varepsilon} \right) \nabla u_\alpha^\varepsilon \right) + \frac{1}{\varepsilon^2} \sum_{\beta=1}^N \Pi_{\alpha\beta} \left(\frac{x}{\varepsilon} \right) u_\beta^\varepsilon = 0,$$

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for $1 \leq \alpha \leq N$, where b_α are velocity fields, D_α are symmetric and coercive diffusion tensors and Π is the reaction (or coupling) matrix, assumed to be cooperative (see (2.7) for a precise definition). All coefficients are Y -periodic, where $Y :=]0, 1[^d$ is the unit cell in \mathbb{R}^d . Our main result, Theorem 18, states that a solution to the Cauchy problem for (1.1) admits the following asymptotic representation (for every $1 \leq \alpha \leq N$):

$$u_\alpha^\varepsilon(t, x) = \varphi_\alpha\left(\frac{x}{\varepsilon}\right) \exp(-\lambda t/\varepsilon^2) \left(v\left(t, x - \frac{b^* t}{\varepsilon}\right) + \mathcal{O}(\varepsilon)\right),$$

where $\{\lambda, (\varphi_\alpha)_{1 \leq \alpha \leq N}\}$ is the first eigenpair for a periodic system posed in the unit cell $Y :=]0, 1[^d$, b^* is a constant drift vector and $v(t, x)$ solves a scalar parabolic homogenized problem with constant coefficients. Our result generalizes the works [7] and [14], which were restricted to a single (scalar) parabolic equation. In [9], [10] a similar result was obtained for a cooperative elliptic system without convective terms. Our present work is thus the first to combine large convective terms and multiple equations.

Let us explain the specific ε -scaling of the coefficients in (1.1), which is not new and is well explained, e.g., in [6]. Before adimensionalization, the physical system of equations, in original time-space coordinates (τ, y) , is, for $1 \leq \alpha \leq N$,

$$\rho_\alpha \frac{\partial u_\alpha}{\partial \tau} + b_\alpha \cdot \nabla u_\alpha - \operatorname{div}(D_\alpha \nabla u_\alpha) + \sum_{\beta=1}^N \Pi_{\alpha\beta} u_\beta = 0.$$

Interested by a macroscopic view and long time behaviour of this parabolic system, we perform a “parabolic” scaling of the time-space variables, i.e., $(\tau, y) \rightarrow (\varepsilon^{-2}t, \varepsilon^{-1}x)$, which yields the scaled model (1.1).

Remark 1. *Another scaling that one could consider is the “hyperbolic” scaling, i.e., $(\tau, y) \rightarrow (\varepsilon^{-1}t, \varepsilon^{-1}x)$. This has been addressed in [21] (for $N = 2$) where the scaled system is:*

$$\rho_\alpha \left(\frac{x}{\varepsilon}\right) \frac{\partial u_\alpha^\varepsilon}{\partial t} + b_\alpha \left(\frac{x}{\varepsilon}\right) \cdot \nabla u_\alpha^\varepsilon - \varepsilon \operatorname{div}\left(D_\alpha \left(\frac{x}{\varepsilon}\right) \nabla u_\alpha^\varepsilon\right) + \frac{1}{\varepsilon} \sum_{\beta=1}^N \Pi_{\alpha\beta} \left(\frac{x}{\varepsilon}\right) u_\beta^\varepsilon = 0,$$

for $1 \leq \alpha \leq N$. The main result of [21] is that the solution to the Cauchy problem for the above system admits the asymptotic representation:

$$u_\alpha^\varepsilon(t, x) \approx \phi_\alpha\left(\frac{x}{\varepsilon}\right) \delta(x - b^* t)$$

where ϕ_α is the first eigenfunction and there is no time exponential because $\lambda = 0$ happens to be the first eigenvalue for the specific choice of cooperative matrix $\Pi_{\alpha\beta}$ made in [21]. In the above equation δ is the Dirac mass which appears because of a concentration assumption on the initial data. The main difference with the parabolic scaling in our work is that there is no diffusion homogenized problem. The drift velocity can be interpreted as $b^* = \nabla H(0)$ with H being some effective Hamiltonian.

The organization of this paper is as follows. In Section 2, we describe the mathematical model of cooperative parabolic systems and the precise hypotheses made on the coefficients. Section 3 briefly recalls the existence and uniqueness theory for system (1.1). Since no uniform a priori estimates can be obtained for (1.1), a factorization principle (or change of unknowns) is performed in Section 4. Then, uniform a priori bounds are deduced for the solution of this factorized problem. The definition of two-scale convergence with drift is recalled in Section 5. Then, based on the uniform a priori estimates of Section 4, we obtain a two-scale compactness result for the sequence of solutions (see Theorem 15). Our main homogenization result is Theorem 18 which is proved in Section 6. Eventually, Section 7 generalizes our analysis to a similar, but more involved, system which is meaningful from a physical point of view. The differences are that (i) the convection-diffusion takes place in a perforated porous medium and (ii) the chemical reactions are localized on the holes' boundaries rather than in the fluid bulk. This is a frequent case for adsorption or deposition of the chemical on the solid surface (cf. the discussion and references in [6]).

2. THE MODEL

Before we present our model, let us introduce the following shorthands:

$$\rho_\alpha^\varepsilon(x) := \rho_\alpha\left(\frac{x}{\varepsilon}\right); \quad b_\alpha^\varepsilon(x) := b_\alpha\left(\frac{x}{\varepsilon}\right); \quad D_\alpha^\varepsilon(x) := D_\alpha\left(\frac{x}{\varepsilon}\right); \quad \Pi_{\alpha\beta}^\varepsilon(x) := \Pi_{\alpha\beta}\left(\frac{x}{\varepsilon}\right),$$

where the small positive parameter $\varepsilon \ll 1$ represents the lengthscale of oscillations. We consider the following Cauchy problem:

$$(2.1) \quad \rho_\alpha^\varepsilon \frac{\partial u_\alpha^\varepsilon}{\partial t} + \frac{1}{\varepsilon} b_\alpha^\varepsilon \cdot \nabla u_\alpha^\varepsilon - \operatorname{div}(D_\alpha^\varepsilon \nabla u_\alpha^\varepsilon) + \frac{1}{\varepsilon^2} \sum_{\beta=1}^N \Pi_{\alpha\beta}^\varepsilon u_\beta^\varepsilon = 0 \quad \text{in } (0, T) \times \mathbb{R}^d,$$

$$(2.2) \quad u_\alpha^\varepsilon(0, x) = u_\alpha^{\text{in}}(x) \quad \text{for } x \in \mathbb{R}^d.$$

For a normed vector space \mathcal{H} , we use the following standard notation for Y -periodic function spaces:

$$L_\#^p(\mathbb{R}^d; \mathcal{H}) := \left\{ f : \mathbb{R}^d \rightarrow \mathcal{H} \text{ s.t. } f \text{ is } Y\text{-periodic and } \|f\|_{\mathcal{H}} \|_{L^p(Y)} < \infty \right\}.$$

The assumptions made on the coefficients of (2.1) are the following:

$$(2.3) \quad \rho_\alpha \in L_\#^\infty(\mathbb{R}^d; \mathbb{R}) \text{ and } \exists c_\alpha > 0 \text{ s.t. } \rho_\alpha(y) \geq c_\alpha,$$

$$(2.4) \quad b_\alpha \in L_\#^\infty(\mathbb{R}^d; \mathbb{R}^d) \text{ and } \operatorname{div} b_\alpha \in L_\#^\infty(\mathbb{R}^d; \mathbb{R}),$$

$$(2.5) \quad D_\alpha = (D_\alpha)^* \in L_\#^\infty(\mathbb{R}^d; \mathbb{R}^{d \times d}) \text{ and } \exists c_\alpha > 0 \text{ s.t. } c_\alpha |\xi|^2 \leq D_\alpha(y) \xi \cdot \xi$$

for all $\xi \in \mathbb{R}^d$ and for almost every $y \in \mathbb{R}^d$ (where $(D_\alpha)^*$ is the adjoint or transposed matrix of D_α),

$$(2.6) \quad \Pi \in L_\#^\infty(\mathbb{R}^d; \mathbb{R}^{d \times d}) \text{ and } \Pi_{\alpha\beta} \leq 0 \text{ for } \alpha \neq \beta,$$

we also assume that the coupling matrix Π is irreducible, i.e., there exists no partition $\mathcal{B} \neq \emptyset, \mathcal{B}' \neq \emptyset$ of $\{1, \dots, N\}$ such that

$$(2.7) \quad \{1, \dots, N\} = \mathcal{B} \cup \mathcal{B}' \text{ with } \mathcal{B} \cap \mathcal{B}' = \emptyset \text{ and } \Pi_{\alpha\beta} = 0 \text{ for all } \alpha \in \mathcal{B}, \beta \in \mathcal{B}'.$$

This irreducibility assumption ensures that the system (2.1) cannot be decoupled in two disjoint subsystems (see Remark 19 below).

Remark 2. *The only assumption made on the convective fields b_α in (2.4) is that they are bounded as well as their divergences. No divergence-free assumption is made on these vector fields. The hypotheses (2.6)-(2.7) are borrowed from [24, 22, 3, 10]. A matrix satisfying (2.6) is sometimes referred to as “cooperative matrix” (up to the addition of a multiple of the identity it is also an M -matrix). Hence the system (2.1) gets the name “cooperative parabolic system”.*

Finally, we assume that the initial data in (2.2) has following regularity: $u_\alpha^{in} \in L^2(\mathbb{R}^d)$ for each $1 \leq \alpha \leq N$.

3. QUALITATIVE ANALYSIS

Results of existence and uniqueness of solutions to (2.1) are classical. The “cooperative” hypothesis (2.6) is actually not necessary to obtain well-posedness. Standard approach is to derive a priori estimates on the solution. Classical technique is to multiply (2.1) by u_α^ε and integrate over \mathbb{R}^d :

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \rho_\alpha^\varepsilon |u_\alpha^\varepsilon|^2 dx + \int_{\mathbb{R}^d} D_\alpha^\varepsilon \nabla u_\alpha^\varepsilon \cdot \nabla u_\alpha^\varepsilon dx \\ &= \frac{1}{2\varepsilon} \int_{\mathbb{R}^d} \operatorname{div}(b_\alpha^\varepsilon) |u_\alpha^\varepsilon|^2 dx - \frac{1}{\varepsilon^2} \sum_{\beta=1}^N \int_{\mathbb{R}^d} \Pi_{\alpha\beta}^\varepsilon u_\beta^\varepsilon u_\alpha^\varepsilon dx. \end{aligned}$$

Since the divergences of the convective fields are bounded, summing the above expression over $1 \leq \alpha \leq N$ followed by the application of Cauchy-Schwarz inequality, Young’s inequality, Gronwall’s lemma and an integration over $(0, T)$ leads to the following a priori estimates:

$$(3.1) \quad \sum_{\alpha=1}^N \|u_\alpha^\varepsilon\|_{L^\infty((0,T);L^2(\mathbb{R}^d))} + \sum_{\alpha=1}^N \|\nabla u_\alpha^\varepsilon\|_{L^2((0,T)\times\mathbb{R}^d)} \leq C_\varepsilon \sum_{\alpha=1}^N \|u_\alpha^{in}\|_{L^2(\mathbb{R}^d)},$$

where the constant C_ε depends on the small parameter ε . For any fixed $0 < \varepsilon$, we can use the a priori estimates (3.1) and Galerkin method to establish existence and uniqueness of the solution $u_\alpha^\varepsilon \in L^2((0, T); H^1(\mathbb{R}^d)) \cap C((0, T); L^2(\mathbb{R}^d))$, $1 \leq \alpha \leq N$.

Maximum principles are a different story altogether. In general we have no maximum principles for systems. However, the hypotheses (2.6)-(2.7) guarantee a maximum principle. In [24, 22], weakly coupled cooperative elliptic systems with coupling matrices satisfying (2.6)-(2.7) are studied with emphasis on maximum principles and on the well-posedness of associated spectral problems. The results

of [24] on the cooperative elliptic systems have their parabolic counterpart. We state the result from [24] adapted to cooperative parabolic systems:

Lemma 3 (see [24, 22] for a proof). *Let the conditions (2.3)-(2.7) on the coefficients of (2.1) be satisfied. Then, for any fixed $\varepsilon > 0$, the following holds:*

(i) *There is a unique solution $u_\alpha^\varepsilon \in L^2((0, T); H^1(\mathbb{R}^d)) \cap C((0, T); L^2(\mathbb{R}^d))$ for $1 \leq \alpha \leq N$.*

(ii) *If $u_\alpha^{in} \geq 0$ for all $1 \leq \alpha \leq N$, then $u_\alpha^\varepsilon \geq 0$ for all $1 \leq \alpha \leq N$.*

Remark 4. *In order to make an asymptotic analysis on (2.1), as $\varepsilon \rightarrow 0$, one demands uniform (with respect to ε) estimates on the solution u_α^ε . But the estimates in (3.1) are not uniform in ε . This renders the application of standard compactness theorems from homogenization theory useless for (2.1).*

4. FACTORIZATION PRINCIPLE

The difficulty with the derivation of a priori estimates in presence of large lower order terms has long been recognized [25, 17, 18, 3, 4, 14]. The idea is to use information from an associated spectral cell problem. The basic principle is to factor out principal eigenfunction from the solution to arrive at a new “factorized system”, amenable to uniform a priori estimates. This idea of factoring our oscillations from the solution was first introduced in [25] in the context of elliptic eigenvalue problems. In case of scalar parabolic equations it is shown in [17, 18, 14, 7] that the factorized equations have no zero order terms and that the first order terms are divergence free. In case of cooperative elliptic systems with large lower order terms studied in [3, 10], however, it is shown that the factorized systems still have zero order terms and are transformed as “difference terms”. We adopt the “factorization principle”, extensively used in the above mentioned references, to remedy the difficulty we have with the derivation of uniform a priori estimates for (2.1). We first define the following spectral problem associated with (2.1) and posed in the unit cell with periodic boundary conditions:

$$(4.1) \quad \begin{cases} b_\alpha \cdot \nabla_y \varphi_\alpha - \operatorname{div}_y (D_\alpha \nabla_y \varphi_\alpha) + \sum_{\beta=1}^N \Pi_{\alpha\beta} \varphi_\beta = \lambda \rho_\alpha \varphi_\alpha & \text{in } Y, \\ y \rightarrow \varphi_\alpha(y) & Y\text{-periodic.} \end{cases}$$

The above spectral cell problem is not self-adjoint. The associated adjoint problem is:

$$(4.2) \quad \begin{cases} -\operatorname{div}_y (b_\alpha \varphi_\alpha^*) - \operatorname{div}_y (D_\alpha \nabla_y \varphi_\alpha^*) + \sum_{\beta=1}^N \Pi_{\alpha\beta}^* \varphi_\beta^* = \lambda \rho_\alpha \varphi_\alpha^* & \text{in } Y, \\ y \rightarrow \varphi_\alpha^*(y) & Y\text{-periodic,} \end{cases}$$

where Π^* is the transpose of Π . The well-posedness of the above spectral problems is a delicate issue which is addressed in [24, 22]. The following proposition is an adaptation to our periodic setting of the main result of [24, 22].

Proposition 5 (see [24] for a proof). *Under the assumptions (2.3)-(2.7) on the coefficients, the spectral problems (4.1) and (4.2) admit a common first eigenvalue (i.e., smallest in modulus) which satisfies:*

- (i) *the first eigenvalue λ is real and simple,*
- (ii) *the corresponding first eigenfunctions $(\varphi_\alpha)_{1 \leq \alpha \leq N} \in (H_\#^1(Y))^N$ for (4.1), $(\varphi_\alpha^*)_{1 \leq \alpha \leq N} \in (H_\#^1(Y))^N$ for (4.2) are positive, $\varphi_\alpha, \varphi_\alpha^* > 0$ for $1 \leq \alpha \leq N$, and unique up to normalization.*

Remark 6. *The first eigenvalue λ in Proposition 5 measures the balance between convection-diffusion and reaction. Also, the uniqueness of first eigenfunctions in Proposition 5 is up to a chosen normalization. The normalization that we consider is the following:*

$$(4.3) \quad \sum_{\alpha=1}^N \int_Y \rho_\alpha \varphi_\alpha \varphi_\alpha^* dy = 1.$$

In the proof of our a priori estimates it will be convenient to scale the spectral problems (4.1)-(4.2) to the entire domain \mathbb{R}^d via the change of variables $y \rightarrow \varepsilon^{-1}x$. More precisely, (4.1)-(4.2) are equivalent to

$$(4.4) \quad \begin{cases} \varepsilon b_\alpha^\varepsilon \cdot \nabla \varphi_\alpha^\varepsilon - \varepsilon^2 \operatorname{div} \left(D_\alpha^\varepsilon \nabla \varphi_\alpha^\varepsilon \right) + \sum_{\beta=1}^N \Pi_{\alpha\beta}^\varepsilon \varphi_\beta^\varepsilon = \lambda \rho_\alpha^\varepsilon \varphi_\alpha^\varepsilon & \text{in } \mathbb{R}^d, \\ x \rightarrow \varphi_\alpha^\varepsilon(x) \equiv \varphi_\alpha(x/\varepsilon) & \varepsilon Y\text{-periodic}, \end{cases}$$

$$(4.5) \quad \begin{cases} -\varepsilon \operatorname{div} (b_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon}) - \varepsilon^2 \operatorname{div} \left(D_\alpha^\varepsilon \nabla \varphi_\alpha^{*\varepsilon} \right) + \sum_{\beta=1}^N \Pi_{\alpha\beta}^{*\varepsilon} \varphi_\beta^{*\varepsilon} = \lambda \rho_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon} & \text{in } \mathbb{R}^d, \\ x \rightarrow \varphi_\alpha^{*\varepsilon}(x) \equiv \varphi_\alpha^*(x/\varepsilon) & \varepsilon Y\text{-periodic}. \end{cases}$$

Now, we get down to the task of reducing (2.1) to a “factorized system”. As explained in [3, 4, 14, 7] the first eigenvalue λ governs the time decay or growth of the solution u_α^ε . So, as is done in the references cited, we perform time renormalization in the spirit of the factorization principle. Also the first eigenfunction $\varphi_\alpha^\varepsilon$ is factored out of u_α^ε . In other words we make the following change of unknowns:

$$(4.6) \quad v_\alpha^\varepsilon(t, x) = \exp(\lambda t / \varepsilon^2) \frac{u_\alpha^\varepsilon(t, x)}{\varphi_\alpha^\varepsilon(x)}.$$

The above change of unknowns is valid, thanks to the positivity result in Proposition 5. Now we state a result that gives the factorized system satisfied by the new unknown $(v_\alpha^\varepsilon)_{1 \leq \alpha \leq N}$.

Lemma 7. *The system (2.1)-(2.2) is equivalent to*

$$(4.7) \quad \varphi_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon} \rho_\alpha^\varepsilon \frac{\partial v_\alpha^\varepsilon}{\partial t} + \frac{1}{\varepsilon} \tilde{b}_\alpha^\varepsilon \cdot \nabla v_\alpha^\varepsilon - \operatorname{div} \left(\tilde{D}_\alpha^\varepsilon \nabla v_\alpha^\varepsilon \right) + \frac{1}{\varepsilon^2} \sum_{\beta=1}^N \Pi_{\alpha\beta}^\varepsilon \varphi_\alpha^{*\varepsilon} \varphi_\beta^\varepsilon (v_\beta^\varepsilon - v_\alpha^\varepsilon) = 0$$

in $(0, T) \times \mathbb{R}^d$ for each $1 \leq \alpha \leq N$ complemented with the initial data:

$$(4.8) \quad v_\alpha^\varepsilon(0, x) = \frac{u_\alpha^{\text{in}}(x)}{\varphi_\alpha^\varepsilon(x)} \quad x \in \mathbb{R}^d,$$

for each $1 \leq \alpha \leq N$, where the components of $(v_\alpha^\varepsilon)_{1 \leq \alpha \leq N}$ are defined by (4.6). The convective velocities, $\tilde{b}_\alpha^\varepsilon(x) = \tilde{b}_\alpha\left(\frac{x}{\varepsilon}\right)$, in (4.7) are given by

$$(4.9) \quad \tilde{b}_\alpha(y) = \varphi_\alpha \varphi_\alpha^* b_\alpha + \varphi_\alpha D_\alpha \nabla_y \varphi_\alpha^* - \varphi_\alpha^* D_\alpha \nabla_y \varphi_\alpha \quad \text{for every } 1 \leq \alpha \leq N$$

and the diffusion matrices, $\tilde{D}_\alpha^\varepsilon(x) = \tilde{D}_\alpha\left(\frac{x}{\varepsilon}\right)$, in (4.7) are given by

$$(4.10) \quad \tilde{D}_\alpha(y) = \varphi_\alpha \varphi_\alpha^* D_\alpha \quad \text{for every } 1 \leq \alpha \leq N.$$

The proof of Lemma 7 is just a matter of simple algebra, using (4.4), and we refer to [10], [16] for more details, keeping in mind the following chain rule formulae:

$$\begin{cases} \frac{\partial u_\alpha^\varepsilon}{\partial t}(t, x) = \exp(-\lambda t/\varepsilon^2) \left(\frac{-\lambda}{\varepsilon^2} \varphi_\alpha\left(\frac{x}{\varepsilon}\right) v_\alpha^\varepsilon(t, x) + \varphi_\alpha\left(\frac{x}{\varepsilon}\right) \frac{\partial v_\alpha^\varepsilon}{\partial t}(t, x) \right), \\ \nabla \left(u_\alpha^\varepsilon(t, x) \right) = \exp(-\lambda t/\varepsilon^2) \left(\frac{1}{\varepsilon} v_\alpha^\varepsilon(t, x) \left(\nabla_y \varphi_\alpha \right) \left(\frac{x}{\varepsilon} \right) + \varphi_\alpha\left(\frac{x}{\varepsilon}\right) \nabla_x v_\alpha^\varepsilon(t, x) \right). \end{cases}$$

Remark 8. *The divergence of the convective fields \tilde{b}_α satisfy*

$$(4.11) \quad \operatorname{div}_y \tilde{b}_\alpha = \sum_{\beta=1}^N \Pi_{\alpha\beta}^* \varphi_\alpha \varphi_\beta^* - \sum_{\beta=1}^N \Pi_{\alpha\beta} \varphi_\alpha^* \varphi_\beta.$$

It follows that

$$\sum_{\alpha=1}^N \operatorname{div}_y \tilde{b}_\alpha = 0.$$

Remark 9. *The factorized system (4.7) still has large lower order terms. But, as noticed in [3, 10], the terms are transformed as “difference terms”. This factorization is the key for getting a priori estimate on the differences $(v_\alpha^\varepsilon - v_\beta^\varepsilon)$.*

The following lemma gives the a priori estimates on the new unknown.

Lemma 10. *Let $(v_\alpha^\varepsilon)_{1 \leq \alpha \leq N}$ be a weak solution of (4.7)-(4.8). There exists a constant C , independent of ε , such that*

$$\begin{aligned}
(4.12) \quad & \sum_{\alpha=1}^N \left\| v_{\alpha}^{\varepsilon} \right\|_{L^{\infty}((0,T);L^2(\mathbb{R}^d))} + \sum_{\alpha=1}^N \left\| \nabla v_{\alpha}^{\varepsilon} \right\|_{L^2((0,T) \times \mathbb{R}^d)} \\
& + \frac{1}{\varepsilon} \sum_{\alpha=1}^N \sum_{\beta=1}^N \left\| v_{\alpha}^{\varepsilon} - v_{\beta}^{\varepsilon} \right\|_{L^2((0,T) \times \mathbb{R}^d)} \leq C \sum_{\alpha=1}^N \|v_{\alpha}^{in}\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Proof. To derive the a priori estimates, we multiply (4.7) by v_{α}^{ε} followed by integrating over \mathbb{R}^d and sum the obtained expressions over $1 \leq \alpha \leq N$:

$$\begin{aligned}
(4.13) \quad & \frac{1}{2} \frac{d}{dt} \sum_{\alpha=1}^N \int_{\mathbb{R}^d} \varphi_{\alpha}^{\varepsilon} \varphi_{\alpha}^{*\varepsilon} \rho_{\alpha}^{\varepsilon} |v_{\alpha}^{\varepsilon}|^2 dx - \frac{1}{2\varepsilon} \sum_{\alpha=1}^N \int_{\mathbb{R}^d} \operatorname{div}(\tilde{b}_{\alpha}^{\varepsilon}) |v_{\alpha}^{\varepsilon}|^2 dx \\
& + \sum_{\alpha=1}^N \int_{\mathbb{R}^d} \tilde{D}_{\alpha}^{\varepsilon} \nabla v_{\alpha}^{\varepsilon} \cdot \nabla v_{\alpha}^{\varepsilon} dx + \frac{1}{\varepsilon^2} \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_{\mathbb{R}^d} \Pi_{\alpha\beta}^{\varepsilon} \varphi_{\alpha}^{*\varepsilon} \varphi_{\beta}^{\varepsilon} (v_{\beta}^{\varepsilon} - v_{\alpha}^{\varepsilon}) v_{\alpha}^{\varepsilon} = 0.
\end{aligned}$$

To simplify the above expressions, we now use the scaled spectral problems (4.4)-(4.5). Multiply (4.4) by $\varphi_{\alpha}^{*\varepsilon} (v_{\alpha}^{\varepsilon})^2$ followed by integration over the space domain \mathbb{R}^d :

$$\begin{aligned}
& \frac{1}{\varepsilon} \int_{\mathbb{R}^d} (b_{\alpha}^{\varepsilon} \cdot \nabla \varphi_{\alpha}^{\varepsilon}) \varphi_{\alpha}^{*\varepsilon} (v_{\alpha}^{\varepsilon})^2 dx - \int_{\mathbb{R}^d} \operatorname{div}(D_{\alpha}^{\varepsilon} \nabla \varphi_{\alpha}^{\varepsilon}) \varphi_{\alpha}^{*\varepsilon} (v_{\alpha}^{\varepsilon})^2 dx \\
& + \frac{1}{\varepsilon^2} \sum_{\beta=1}^N \int_{\mathbb{R}^d} \Pi_{\alpha\beta}^{\varepsilon} \varphi_{\beta}^{\varepsilon} \varphi_{\alpha}^{*\varepsilon} (v_{\alpha}^{\varepsilon})^2 dx - \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} \lambda \rho_{\alpha}^{\varepsilon} \varphi_{\alpha}^{\varepsilon} \varphi_{\alpha}^{*\varepsilon} (v_{\alpha}^{\varepsilon})^2 dx \\
& = -\frac{1}{\varepsilon} \int_{\mathbb{R}^d} \operatorname{div}(b_{\alpha}^{\varepsilon} \varphi_{\alpha}^{*\varepsilon}) \varphi_{\alpha}^{\varepsilon} (v_{\alpha}^{\varepsilon})^2 dx - \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \varphi_{\alpha}^{\varepsilon} \varphi_{\alpha}^{*\varepsilon} b_{\alpha}^{\varepsilon} \cdot \nabla (v_{\alpha}^{\varepsilon})^2 dx \\
& + \int_{\mathbb{R}^d} \varphi_{\alpha}^{*\varepsilon} D_{\alpha}^{\varepsilon} \nabla \varphi_{\alpha}^{\varepsilon} \cdot \nabla (v_{\alpha}^{\varepsilon})^2 dx + \int_{\mathbb{R}^d} (v_{\alpha}^{\varepsilon})^2 D_{\alpha}^{\varepsilon} \nabla \varphi_{\alpha}^{\varepsilon} \cdot \nabla \varphi_{\alpha}^{*\varepsilon} dx \\
& + \frac{1}{\varepsilon^2} \sum_{\beta=1}^N \int_{\mathbb{R}^d} \Pi_{\alpha\beta}^{\varepsilon} \varphi_{\alpha}^{*\varepsilon} \varphi_{\beta}^{\varepsilon} (v_{\alpha}^{\varepsilon})^2 dx - \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} \lambda \rho_{\alpha}^{\varepsilon} \varphi_{\alpha}^{\varepsilon} \varphi_{\alpha}^{*\varepsilon} (v_{\alpha}^{\varepsilon})^2 dx \\
& = -\frac{1}{\varepsilon} \int_{\mathbb{R}^d} \operatorname{div}(b_{\alpha}^{\varepsilon} \varphi_{\alpha}^{*\varepsilon}) \varphi_{\alpha}^{\varepsilon} (v_{\alpha}^{\varepsilon})^2 dx - \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \varphi_{\alpha}^{\varepsilon} \varphi_{\alpha}^{*\varepsilon} b_{\alpha}^{\varepsilon} \cdot \nabla (v_{\alpha}^{\varepsilon})^2 dx \\
& + \int_{\mathbb{R}^d} \varphi_{\alpha}^{*\varepsilon} D_{\alpha}^{\varepsilon} \nabla \varphi_{\alpha}^{\varepsilon} \cdot \nabla (v_{\alpha}^{\varepsilon})^2 dx + \int_{\mathbb{R}^d} (v_{\alpha}^{\varepsilon})^2 D_{\alpha}^{\varepsilon} \nabla \varphi_{\alpha}^{\varepsilon} \cdot \nabla \varphi_{\alpha}^{*\varepsilon} dx \\
& - \int_{\mathbb{R}^d} \varphi_{\alpha}^{\varepsilon} D_{\alpha}^{\varepsilon} \nabla \varphi_{\alpha}^{*\varepsilon} \cdot \nabla (v_{\alpha}^{\varepsilon})^2 dx + \int_{\mathbb{R}^d} \varphi_{\alpha}^{\varepsilon} D_{\alpha}^{\varepsilon} \nabla \varphi_{\alpha}^{*\varepsilon} \cdot \nabla (v_{\alpha}^{\varepsilon})^2 dx \\
& + \frac{1}{\varepsilon^2} \sum_{\beta=1}^N \int_{\mathbb{R}^d} \Pi_{\alpha\beta}^{\varepsilon} \varphi_{\alpha}^{*\varepsilon} \varphi_{\beta}^{\varepsilon} (v_{\alpha}^{\varepsilon})^2 dx - \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} \lambda \rho_{\alpha}^{\varepsilon} \varphi_{\alpha}^{\varepsilon} \varphi_{\alpha}^{*\varepsilon} (v_{\alpha}^{\varepsilon})^2 dx \\
& = -\frac{1}{\varepsilon} \int_{\mathbb{R}^d} \operatorname{div}(b_{\alpha}^{\varepsilon} \varphi_{\alpha}^{*\varepsilon}) \varphi_{\alpha}^{\varepsilon} (v_{\alpha}^{\varepsilon})^2 dx - \int_{\mathbb{R}^d} \operatorname{div}(D_{\alpha}^{\varepsilon} \nabla \varphi_{\alpha}^{*\varepsilon}) \varphi_{\alpha}^{\varepsilon} (v_{\alpha}^{\varepsilon})^2 dx
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} \lambda \rho_\alpha^\varepsilon \varphi_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon} (v_\alpha^\varepsilon)^2 dx - \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \varphi_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon} b_\alpha^\varepsilon \cdot \nabla (v_\alpha^\varepsilon)^2 dx \\
& + \int_{\mathbb{R}^d} \varphi_\alpha^{*\varepsilon} D_\alpha^\varepsilon \nabla \varphi_\alpha^\varepsilon \cdot \nabla (v_\alpha^\varepsilon)^2 dx - \int_{\mathbb{R}^d} \varphi_\alpha^\varepsilon D_\alpha^\varepsilon \nabla \varphi_\alpha^{*\varepsilon} \cdot \nabla (v_\alpha^\varepsilon)^2 dx \\
& + \frac{1}{\varepsilon^2} \sum_{\beta=1}^N \int_{\mathbb{R}^d} \Pi_{\alpha\beta}^\varepsilon \varphi_\alpha^{*\varepsilon} \varphi_\beta^\varepsilon (v_\alpha^\varepsilon)^2 dx = 0.
\end{aligned}$$

In the above expression, we recognize the scaled adjoint cell problem (4.5). We also recognize the scaled expression of (4.9) for the convective field \tilde{b}_α . Taking all these into consideration, we have the following:

$$-\frac{1}{\varepsilon} \int_{\mathbb{R}^d} \tilde{b}_\alpha^\varepsilon \cdot \nabla (v_\alpha^\varepsilon)^2 dx + \frac{1}{\varepsilon^2} \sum_{\beta=1}^N \int_{\mathbb{R}^d} \left(\Pi_{\alpha\beta}^\varepsilon \varphi_\alpha^{*\varepsilon} \varphi_\beta^\varepsilon - \Pi_{\alpha\beta}^{*\varepsilon} \varphi_\beta^{*\varepsilon} \varphi_\alpha^\varepsilon \right) (v_\alpha^\varepsilon)^2 dx = 0.$$

Summing over α , we have:

$$\begin{aligned}
(4.14) \quad & -\frac{1}{2\varepsilon} \sum_{\alpha=1}^N \int_{\mathbb{R}^d} \operatorname{div}(\tilde{b}_\alpha^\varepsilon) (v_\alpha^\varepsilon)^2 dx = \frac{1}{2\varepsilon^2} \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_{\mathbb{R}^d} \left(\Pi_{\alpha\beta}^\varepsilon \varphi_\alpha^{*\varepsilon} \varphi_\beta^\varepsilon - \Pi_{\alpha\beta}^{*\varepsilon} \varphi_\beta^{*\varepsilon} \varphi_\alpha^\varepsilon \right) (v_\alpha^\varepsilon)^2 dx.
\end{aligned}$$

Now, let us employ (4.14) in the estimate (4.13) which leads to:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \sum_{\alpha=1}^N \int_{\mathbb{R}^d} \varphi_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon} \rho_\alpha^\varepsilon |v_\alpha^\varepsilon|^2 dx + \sum_{\alpha=1}^N \int_{\mathbb{R}^d} \tilde{D}_\alpha^\varepsilon \nabla v_\alpha^\varepsilon \cdot \nabla v_\alpha^\varepsilon dx \\
& + \frac{1}{\varepsilon^2} \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_{\mathbb{R}^d} \left\{ \Pi_{\alpha\beta}^\varepsilon \varphi_\alpha^{*\varepsilon} \varphi_\beta^\varepsilon \left(v_\beta^\varepsilon v_\alpha^\varepsilon - \frac{1}{2} (v_\alpha^\varepsilon)^2 \right) - \frac{1}{2} \Pi_{\alpha\beta}^{*\varepsilon} \varphi_\beta^{*\varepsilon} \varphi_\alpha^\varepsilon (v_\alpha^\varepsilon)^2 \right\} dx = 0.
\end{aligned}$$

The above expression is nothing but the following energy estimate:

$$\begin{aligned}
(4.15) \quad & \frac{1}{2} \frac{d}{dt} \sum_{\alpha=1}^N \int_{\mathbb{R}^d} \varphi_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon} \rho_\alpha^\varepsilon |v_\alpha^\varepsilon|^2 dx + \sum_{\alpha=1}^N \int_{\mathbb{R}^d} \tilde{D}_\alpha^\varepsilon \nabla v_\alpha^\varepsilon \cdot \nabla v_\alpha^\varepsilon dx \\
& - \frac{1}{2\varepsilon^2} \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_{\mathbb{R}^d} \Pi_{\alpha\beta}^\varepsilon \varphi_\alpha^{*\varepsilon} \varphi_\beta^\varepsilon |v_\alpha^\varepsilon - v_\beta^\varepsilon|^2 dx = 0.
\end{aligned}$$

Each one of the integrands in the above estimate is positive because of the positivity assumption (2.3), coercivity assumption (2.5) and the cooperative assumption (2.6). Integrating the energy estimate (4.15) over $(0, T)$ yields the a priori estimates (4.12). \square

5. TWO-SCALE COMPACTNESS

The homogenization procedure is to consider the weak formulation of (4.7)-(4.8) with appropriately chosen test functions and passing to the limit as $\varepsilon \rightarrow 0$. The usual approach is to obtain two-scale limits using a priori estimates of Lemma 10 by employing some compactness theorems. As it has been noticed in [20, 14, 7], the classical notion of two-scale convergence from [1, 23] needs to be modified in order to address the homogenization of parabolic problems in strong convection regime. We recall this modified notion of two-scale convergence with drift, as first defined in [20].

Definition 11. Let $b^* \in \mathbb{R}^d$ be a constant vector. A sequence of functions $u_\varepsilon(t, x)$ in $L^2((0, T) \times \mathbb{R}^d)$ is said to two-scale converge with drift b^* , or equivalently in moving coordinates $(t, x) \rightarrow (t, x - \frac{b^* t}{\varepsilon})$, to a limit $u_0(t, x, y) \in L^2((0, T) \times \mathbb{R}^d \times Y)$ if, for any function $\phi(t, x, y) \in C_c^\infty((0, T) \times \mathbb{R}^d; C_\#^\infty(Y))$, we have

$$(5.1) \quad \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{R}^d} u_\varepsilon(t, x) \phi\left(t, x - \frac{b^* t}{\varepsilon}, \frac{x}{\varepsilon}\right) dx dt = \int_0^T \int_{\mathbb{R}^d} \int_Y u_0(t, x, y) \phi(t, x, y) dy dx dt.$$

We denote this convergence by $u_\varepsilon \xrightarrow{2\text{-drift}} u_0$.

Now we state a compactness theorem, again borrowed from [20], which guarantees the existence of two-scale limits with drift for certain sequences.

Proposition 12. [20, 2] *Let b^* be a constant vector in \mathbb{R}^d and let the sequence u_ε be uniformly bounded in $L^2((0, T) \times \mathbb{R}^d)$. Then, there exist a subsequence, still denoted by ε , and a function $u_0(t, x, y) \in L^2((0, T) \times \mathbb{R}^d; L_\#^2(Y))$ such that*

$$u_\varepsilon \xrightarrow{2\text{-drift}} u_0.$$

Remark 13. *Note that the case $b^* = 0$ coincides with the classical notion of two-scale convergence from [1, 23]. It should also be noted that the two-scale limits obtained according to Proposition 12 depend on the chosen drift velocity $b^* \in \mathbb{R}^d$. These issues are addressed in [16]. Unfortunately, the notion of convergence in Definition 11 does not carry over to the case when the drift velocity b^* varies in space.*

If the sequence $\{u_\varepsilon\}$ has additional bounds, then the result of Proposition 12 can be improved. The following result addresses this issue when the sequence has uniform H^1 bounds in space.

Proposition 14. [20, 2] *Let b^* be a constant vector in \mathbb{R}^d and let the sequence u_ε be uniformly bounded in $L^2((0, T); H^1(\mathbb{R}^d))$. Then, there exist a subsequence, still denoted by ε , and functions $u_0(t, x) \in L^2((0, T); H^1(\mathbb{R}^d))$ and $u_1(t, x, y) \in L^2((0, T) \times \mathbb{R}^d; H_\#^1(Y))$ such that*

$$u_\varepsilon \xrightarrow{2\text{-drift}} u_0$$

and

$$\nabla u_\varepsilon \xrightarrow{2\text{-drift}} \nabla_x u_0 + \nabla_y u_1.$$

Having given the notion of convergence, we shall state a result that gives the two-scale limits corresponding to solution sequences for (4.7)-(4.8).

Theorem 15. *Let $b^* \in \mathbb{R}^d$ be a constant vector. There exist $v \in L^2((0, T); H^1(\mathbb{R}^d))$ and $v_{1,\alpha} \in L^2((0, T) \times \mathbb{R}^d; H_\#^1(Y))$, for each $1 \leq \alpha \leq N$, such that a subsequence of solutions $(v_\alpha^\varepsilon)_{1 \leq \alpha \leq N} \in L^2((0, T); H^1(\mathbb{R}^d))^N$ of the system (4.7)-(4.8), two-scale converge with drift b^* , as $\varepsilon \rightarrow 0$, in the following sense:*

$$(5.2) \quad \begin{aligned} v_\alpha^\varepsilon &\xrightarrow{2\text{-drift}} v, & \nabla v_\alpha^\varepsilon &\xrightarrow{2\text{-drift}} \nabla_x v + \nabla_y v_{1,\alpha}, \\ & & \frac{1}{\varepsilon}(v_\alpha^\varepsilon - v_\beta^\varepsilon) &\xrightarrow{2\text{-drift}} v_{1,\alpha} - v_{1,\beta}, \end{aligned}$$

for every $1 \leq \alpha, \beta \leq N$.

Proof. Consider the a priori bounds (4.12) on v_α^ε obtained in Lemma 10. It follows from Proposition 14 that there exist a subsequence (still indexed by ε) and two-scale limits, say $v_\alpha \in L^2((0, T); H^1(\mathbb{R}^d))$ and $v_{1,\alpha} \in L^2((0, T) \times \mathbb{R}^d; H_\#^1(Y))$ such that

$$(5.3) \quad \begin{aligned} v_\alpha^\varepsilon &\xrightarrow{2\text{-drift}} v_\alpha \\ \nabla v_\alpha^\varepsilon &\xrightarrow{2\text{-drift}} \nabla_x v_\alpha + \nabla_y v_{1,\alpha} \end{aligned}$$

for every $1 \leq \alpha \leq N$. Also from the a priori estimates (4.12) we have:

$$(5.4) \quad \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_0^T \int_{\mathbb{R}^d} (v_\alpha^\varepsilon - v_\beta^\varepsilon)^2 dx dt \leq C\varepsilon^2.$$

The estimate (5.4) implies that the two-scale limits obtained in the first line of (5.3) do match i.e., $v_\alpha = v$ for every $1 \leq \alpha \leq N$. However, the limit of the coupled term isn't straightforward. Since $\frac{1}{\varepsilon}(v_\alpha^\varepsilon - v_\beta^\varepsilon)$ is bounded in $L^2((0, T) \times \mathbb{R}^d)$, we have the existence of a subsequence and a function $q(t, x, y) \in L^2((0, T) \times \mathbb{R}^d; L_\#^2(Y))$ from Proposition 12 such that

$$(5.5) \quad \frac{1}{\varepsilon}(v_\alpha^\varepsilon - v_\beta^\varepsilon) \xrightarrow{2\text{-drift}} q(t, x, y).$$

Taking $\Psi \in L^2((0, T) \times \mathbb{R}^d \times Y)^d$, let us consider

$$(5.6) \quad \begin{aligned} &\int_0^T \int_{\mathbb{R}^d} (\nabla v_\alpha^\varepsilon - \nabla v_\beta^\varepsilon) \cdot \Psi\left(t, x - \frac{b^*}{\varepsilon}t, \frac{x}{\varepsilon}\right) dx dt = \\ &- \int_0^T \int_{\mathbb{R}^d} (v_\alpha^\varepsilon - v_\beta^\varepsilon) \operatorname{div}_x \Psi\left(t, x - \frac{b^*}{\varepsilon}t, \frac{x}{\varepsilon}\right) dx dt \\ &- \int_0^T \int_{\mathbb{R}^d} \frac{1}{\varepsilon}(v_\alpha^\varepsilon - v_\beta^\varepsilon) \operatorname{div}_y \Psi\left(t, x - \frac{b^*}{\varepsilon}t, \frac{x}{\varepsilon}\right) dx dt. \end{aligned}$$

Let us pass to the limit in (5.6) as $\varepsilon \rightarrow 0$. The first term on the right hand side vanishes as the limits of v_α^ε and v_β^ε match. To pass to the limit in the second term of the right hand side, we shall use (5.5). Considering the two-scale limit in the second line of (5.3), upon passing to the limit as $\varepsilon \rightarrow 0$ in (5.6) we have:

$$(5.7) \quad \begin{aligned} & \int_0^T \int_{\mathbb{R}^d} \int_Y \nabla_y (v_{1,\alpha} - v_{1,\beta}) \cdot \Psi(t, x, y) dy dx dt = \\ & - \int_0^T \int_{\mathbb{R}^d} \int_Y q(t, x, y) \operatorname{div}_y \Psi(t, x, y) dy dx dt. \end{aligned}$$

From (5.7) we deduce that $(v_{1,\alpha} - v_{1,\beta})$ and $q(t, x, y)$ differ by a function of (t, x) , say $l(t, x)$. As $v_{1,\alpha}$ and $v_{1,\beta}$ are also defined up to the addition of a function solely dependent on (t, x) , we can get rid of $l(t, x)$ and we recover indeed the following limit $q(t, x, y) = v_{1,\alpha} - v_{1,\beta}$. \square

6. HOMOGENIZATION RESULT

This section deals with the homogenization of the coupled system (4.7)-(4.8). To begin with, we state a Fredholm alternative for solving the cell problem, which is a key ingredient in the homogenization result.

Lemma 16. *Let $(f_\alpha)_{1 \leq \alpha \leq N} \in (L_\#^2(Y))^N$. Consider the following cooperative system:*

$$(6.1) \quad \begin{cases} \tilde{b}_\alpha \cdot \nabla_y \zeta_\alpha - \operatorname{div}_y (\tilde{D}_\alpha \nabla_y \zeta_\alpha) + \sum_{\beta=1}^N \Pi_{\alpha\beta} \varphi_\alpha^* \varphi_\beta (\zeta_\beta - \zeta_\alpha) = f_\alpha & \text{in } Y, \\ y \rightarrow \zeta_\alpha(y) & Y\text{-periodic,} \end{cases}$$

for every $1 \leq \alpha \leq N$, where the coefficients $(\tilde{b}_\alpha, \tilde{D}_\alpha)$ are as in (4.9)-(4.10) and the hypotheses (2.4)-(2.7) hold. Then there exists a unique solution $(\zeta_\alpha)_{1 \leq \alpha \leq N} \in (H_\#^1(Y))^N / (\mathbb{R} \times \mathbf{1})$ to (6.1), where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^N$, if and only if the following compatibility condition holds true:

$$(6.2) \quad \sum_{\alpha=1}^N \int_Y f_\alpha dy = 0.$$

Proof. To prove that condition (6.2) is necessary, let us integrate the left hand side of (6.1) over the unit cell. Exploiting the periodic boundary conditions, we will be left with:

$$- \int_Y \operatorname{div}_y (\tilde{b}_\alpha) \zeta_\alpha dy + \sum_{\beta=1}^N \int_Y \Pi_{\alpha\beta} \varphi_\alpha^* \varphi_\beta (\zeta_\beta - \zeta_\alpha) dy.$$

Substituting for the divergence term in the above expression from (4.11) and summing over α indeed guarantees that the condition (6.2) on the source term is necessary.

To prove sufficiency, let us assume that (6.2) is satisfied. Consider the following norm on the quotient space $\mathcal{H}(Y) := (H_{\#}^1(Y))^N / (\mathbb{R} \times \mathbb{1})$:

$$(6.3) \quad \|(z_{\alpha})_{1 \leq \alpha \leq N}\|_{\mathcal{H}(Y)}^2 = \sum_{\alpha=1}^N \|\nabla_y z_{\alpha}\|_{L^2(Y)}^2 + \sum_{\alpha=1}^N \sum_{\beta=1}^N \|z_{\alpha} - z_{\beta}\|_{L^2(Y)}^2.$$

(It is easy to show that (6.3) is a norm on $\mathcal{H}(Y)$ since the zero set of (6.3) is the subspace spanned by $\mathbb{1}$.) The variational formulation of (6.1) in $\mathcal{H}(Y)$ is: find $\zeta = (\zeta_{\alpha})_{1 \leq \alpha \leq N} \in \mathcal{H}(Y)$ such that

$$(6.4) \quad \int_Y Q(\zeta) \cdot \eta \, dy = L(\eta) \text{ for any } \eta = (\eta_{\alpha})_{1 \leq \alpha \leq N} \in \mathcal{H}(Y),$$

with

$$\begin{aligned} \int_Y Q(\zeta) \cdot \eta \, dy &:= \sum_{\alpha=1}^N \int_Y \left(\tilde{b}_{\alpha}(y) \cdot \nabla_y \zeta_{\alpha} \right) \eta_{\alpha} \, dy + \sum_{\alpha=1}^N \int_Y \tilde{D}_{\alpha}(y) \nabla_y \zeta_{\alpha} \cdot \nabla_y \eta_{\alpha} \, dy \\ &\quad + \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_Y \Pi_{\alpha\beta} \varphi_{\alpha}^* \varphi_{\beta} (\zeta_{\beta} - \zeta_{\alpha}) \eta_{\alpha} \, dy \end{aligned}$$

and

$$L(\eta) := \sum_{\alpha=1}^N \int_Y f_{\alpha} \eta_{\alpha} \, dy.$$

The compatibility condition (6.2) implies that $(f_{\alpha})_{1 \leq \alpha \leq N}$ is orthogonal to $\mathbb{1}$ in L^2 and consequently that the linear form $L(\eta)$ in (6.4) is continuous.

By performing similar computations as in the proof of Lemma 10, we can show that the bilinear form in (6.4) is coercive in $\mathcal{H}(Y)$ i.e.,

$$\int_Y Q(\zeta) \cdot \zeta \, dy \geq C \sum_{\alpha=1}^N \int_Y |\nabla_y \zeta_{\alpha}|^2 \, dy + \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_Y |\zeta_{\alpha} - \zeta_{\beta}|^2 \, dy.$$

To show that the bilinear form in (6.4) is continuous on $\mathcal{H}(Y) \times \mathcal{H}(Y)$ we remark that, first, $\int_Y Q(\eta) \cdot \mathbb{1} \, dy = 0$ for any $\eta \in \mathcal{H}(Y)$ (this is precisely the computation which yields the compatibility condition (6.2)) and, second, $Q(\eta - c\mathbb{1}) = 0$ for any $\eta \in \mathcal{H}(Y)$ and any $c \in \mathbb{R}$. Therefore, for any $\zeta, \eta \in \mathcal{H}(Y)$, we have the following:

$$\int_Y Q(\zeta) \cdot \eta \, dy = \int_Y Q(\zeta - \mathbb{1}c_{\zeta}) \cdot (\eta - \mathbb{1}c_{\eta}) \, dy \quad \text{for any constants } c_{\zeta}, c_{\eta} \in \mathbb{R},$$

which implies

$$\left| \int_Y Q(\zeta) \cdot \eta \, dy \right| \leq C \left\| (\zeta - \mathbb{1}c_{\zeta}) \right\|_{(H_{\#}^1(Y))^N} \left\| (\eta - \mathbb{1}c_{\eta}) \right\|_{(H_{\#}^1(Y))^N} = C \|\zeta\|_{\mathcal{H}(Y)} \|\eta\|_{\mathcal{H}(Y)}.$$

We can thus apply the Lax-Milgram lemma in $\mathcal{H}(Y)$ to obtain the existence and uniqueness of a solution to (6.1). \square

Remark 17. *The well-posedness result of the Lemma 16 is given in the quotient space $(H_{\#}^1(Y))^N/(\mathbb{R} \times \mathbb{1})$ i.e., the solutions are unique up to the addition of a constant. The constant being the same for each component of the solution.*

In the previous section, using the a priori estimates, we have obtained two-scale limits with drift for the solution sequence. Now, by choosing an appropriate drift constant b^* , we shall characterize the two scale limits. Contrary to the compactness result of Theorem 15 which gives the convergence up to a subsequence, the next result guarantees that the entire sequence v_{α}^{ε} converges to v for every $1 \leq \alpha \leq N$. The main result of this article is the following.

Theorem 18. *Let $(v_{\alpha}^{\varepsilon})_{1 \leq \alpha \leq N}$ be the sequence of solutions to the system (4.7)-(4.8). The entire sequence v_{α}^{ε} converges, in the sense of Theorem 15, to the limits $v \in L^2((0, T); H^1(\mathbb{R}^d))$ and $v_{1, \alpha} \in L^2((0, T) \times \mathbb{R}^d; H_{\#}^1(Y))$ for every $1 \leq \alpha \leq N$ (see (5.2) for details). The two-scale limits $v_{1, \alpha}$ are explicitly given by*

$$(6.5) \quad v_{1, \alpha}(t, x, y) = \sum_{i=1}^d \frac{\partial v}{\partial x_i}(t, x) \omega_{i, \alpha}(y) \quad \text{for every } 1 \leq \alpha \leq N,$$

where $(\omega_{i, \alpha})_{1 \leq \alpha \leq N} \in (H_{\#}^1(Y))^N/(\mathbb{R} \times \mathbb{1})$ satisfy the cell problem:

$$(6.6) \quad \begin{cases} \tilde{b}_{\alpha}(y) \cdot (\nabla_y \omega_{i, \alpha} + e_i) - \operatorname{div}_y (\tilde{D}_{\alpha} (\nabla_y \omega_{i, \alpha} + e_i)) \\ + \sum_{\beta=1}^N \Pi_{\alpha\beta} \varphi_{\alpha}^* \varphi_{\beta} (\omega_{i, \beta} - \omega_{i, \alpha}) = \varphi_{\alpha} \varphi_{\alpha}^* \rho_{\alpha} b^* \cdot e_i & \text{in } Y, \\ y \rightarrow \omega_{i, \alpha}(y) & Y\text{-periodic,} \end{cases}$$

for every $1 \leq i \leq d$, where the drift velocity b^* is given by

$$(6.7) \quad b^* = \sum_{\alpha=1}^N \int_Y \tilde{b}_{\alpha}(y) dy.$$

Further, the two-scale limit $v(t, x)$ is the unique solution of the scalar diffusion equation:

$$(6.8) \quad \begin{cases} \frac{\partial v}{\partial t} - \operatorname{div}(\mathcal{D} \nabla v) = 0 & \text{in } (0, T) \times \mathbb{R}^d, \\ v(0, x) = \sum_{\alpha=1}^N u_{\alpha}^{\text{in}}(x) \int_Y \rho_{\alpha}(y) \varphi_{\alpha}^*(y) dy & \text{in } \mathbb{R}^d, \end{cases}$$

with the elements of the dispersion matrix \mathcal{D} given by

$$(6.9) \quad \begin{aligned} \mathcal{D}_{ij} &= \sum_{\alpha=1}^N \int_Y \tilde{D}_{\alpha} (\nabla_y \omega_{i, \alpha} + e_i) \cdot (\nabla_y \omega_{j, \alpha} + e_j) dy \\ &\quad - \frac{1}{2} \sum_{\alpha, \beta=1}^N \int_Y \varphi_{\alpha}^* \varphi_{\beta} \Pi_{\alpha\beta} (\omega_{i, \alpha} - \omega_{i, \beta}) (\omega_{j, \alpha} - \omega_{j, \beta}) dy. \end{aligned}$$

Remark 19. *The irreducibility assumption (2.7) on the coupling matrix Π ensures microscopic equilibrium among all v_α^ε resulting in a single homogenized limit $v(t, x)$ i.e., if the coupling matrix $\Pi \equiv 0$ (say), we get N different homogenized limits.*

Remark 20. *Our main homogenization result (Theorem 18) holds only for weakly coupled cooperative parabolic systems. Our approach does not answer the homogenization of general weakly coupled parabolic systems, not to mention fully coupled systems. We heavily rely upon the cooperative assumption on the coupling matrix as the positivity and spectral theorems are known only in the cooperative case.*

Remark 21. *The homogenized limit $v(t, x)$ is proven to satisfy a scalar diffusion equation (6.8), which is a bit deceptive by its simplicity. However, if we make the following change of functions:*

$$\tilde{v}(t, x) = \exp(-\lambda t/\varepsilon^2) v\left(t, x - \frac{b^*}{\varepsilon} t\right),$$

we remark that $\tilde{v}(t, x)$ indeed satisfies the following scalar convection-diffusion-reaction equation:

$$\frac{\partial \tilde{v}}{\partial t} + \frac{b^*}{\varepsilon} \cdot \nabla \tilde{v} - \operatorname{div}(\mathcal{D} \nabla \tilde{v}) + \frac{\lambda}{\varepsilon^2} \tilde{v} = 0 \text{ in } (0, T) \times \mathbb{R}^d.$$

Therefore, b^/ε is precisely the effective drift while λ/ε^2 is the effective reaction rate. Remark that because of the large drift $\varepsilon^{-1}b^*$, we cannot work in bounded domains.*

Remark 22. *The assumption of pure periodicity on the coefficients of (2.1) is crucial for the results obtained in this article. The natural thought for generalizing the results of this article is to explore the possibility of considering “locally periodic” coefficients i.e., coefficients of the type $b(x, x/\varepsilon)$, where the function is Y -periodic in the second variable. If the convective fields b_α^ε were locally periodic, then it is clear that the drift vector $b^*(x)$ should depend on x . However, in such a case, we have no idea on how to extend the method of two-scale asymptotic expansion, not to mention the even greater difficulties in generalizing the notion of two-scale convergence with non-constant drift (as already mentioned in Remark 13). Such a generalization still remains as an outstanding open problem in the theory of Taylor dispersion.*

Remark 23. *This article only addresses the homogenization of linear systems. We have also considered only diagonal diffusion models. Cross diffusion phenomena occurs naturally in the physics of multicomponent gaseous mixtures, population dynamics and porous media (cf. [8] and references therein). The natural nonlinear transport model to consider is the Maxwell-Stefan’s equations. A complete mathematical study of the Maxwell-Stefan laws is still missing. There have been some recent studies in this direction (cf. [8, 12, 19, 13] for example). One approach would be to consider the “parabolically” scaled Maxwell-Stefan’s equations and arrive at an homogenization result. The obvious questions to ask is*

the following: Is there a scalar diffusion limit even in case of nonlinear Maxwell-Stefan's equations? This problem might involve mathematical techniques quite different from the ones used here as the spectral problems (which is the crux of the Factorization method) in the nonlinear counterpart have not been well understood. We hope to return to this question in subsequent publications.

Before we present the proof of Theorem 18, we state a lemma that gives some qualitative information on the dispersion matrix.

Lemma 24. *The dispersion matrix \mathcal{D} given by (6.9) is symmetric positive definite.*

Proof. The symmetric part is obvious. By the hypothesis on the coupling matrix Π and the positivity of the first eigenvector functions, the factor $\Pi_{\alpha\beta}\varphi_\alpha^*\varphi_\beta$ is always non-positive for $\alpha \neq \beta$. By the hypothesis (2.5), we know that the diffusion matrices D_α are coercive with coercivity constants $c_\alpha > 0$. For $\xi \in \mathbb{R}^d$, we define

$$\omega_{\alpha\xi} := \sum_{i=1}^d \omega_{i,\alpha} \xi_i.$$

Then,

$$\mathcal{D}\xi \cdot \xi \geq \sum_{\alpha=1}^N c_\alpha \int_Y \left| \nabla_y \omega_{\alpha\xi} + \xi \right|^2 dy \geq 0.$$

Now, we need to show that $\mathcal{D}\xi \cdot \xi > 0$ for all $\xi \neq 0$. Suppose that $\mathcal{D}\xi \cdot \xi = 0$ which in turn implies that $\omega_{\alpha\xi} + \xi \cdot y \equiv C_\alpha$ for some constant C_α . As the cell solutions $(\omega_{i,\alpha})_{1 \leq \alpha \leq N}$ are Y -periodic, they cannot be affine. Thus the above equalities are possible only when $\xi = 0$ which implies the positive definiteness of \mathcal{D} . \square

Proof of Theorem 18. In the sequel we use the notations

$$\begin{aligned} \phi &\equiv \phi(t, x), \quad \phi^\varepsilon \equiv \phi\left(t, x - \frac{b^*t}{\varepsilon}\right), \\ \phi_{1,\alpha} &\equiv \phi_{1,\alpha}(t, x, y), \quad \phi_{1,\alpha}^\varepsilon \equiv \phi_{1,\alpha}\left(t, x - \frac{b^*t}{\varepsilon}, \frac{x}{\varepsilon}\right). \end{aligned}$$

The idea is to test the factorized equation (4.7) with

$$\phi_\alpha^\varepsilon = \phi^\varepsilon + \varepsilon \phi_{1,\alpha}^\varepsilon,$$

where $\phi(t, x)$ and $\phi_{1,\alpha}(t, x, y)$ are smooth functions with compact support in x , which vanish at the final time T and are Y -periodic with respect to y . We get

$$\begin{aligned} &\sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \varphi_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon} \rho_\alpha^\varepsilon \frac{\partial v_\alpha^\varepsilon}{\partial t} \phi_\alpha^\varepsilon dx dt + \frac{1}{\varepsilon} \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \tilde{b}_\alpha^\varepsilon \cdot \nabla v_\alpha^\varepsilon \phi_\alpha^\varepsilon dx dt \\ &+ \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \tilde{D}_\alpha^\varepsilon \nabla v_\alpha^\varepsilon \cdot \nabla \phi_\alpha^\varepsilon dx dt + \frac{1}{\varepsilon^2} \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_0^T \int_{\mathbb{R}^d} \Pi_{\alpha\beta}^\varepsilon \varphi_\alpha^{*\varepsilon} \varphi_\beta^\varepsilon (v_\beta^\varepsilon - v_\alpha^\varepsilon) \phi_\alpha^\varepsilon = 0. \end{aligned}$$

Substituting for ϕ_α^ε in the above variational formulation and integrating by parts leads to

$$\begin{aligned}
(6.10) \quad & - \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \varphi_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon} \rho_\alpha^\varepsilon v_\alpha^\varepsilon \frac{\partial \phi^\varepsilon}{\partial t} dx dt - \sum_{\alpha=1}^N \int_{\mathbb{R}^d} \varphi_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon} \rho_\alpha^\varepsilon v_\alpha^\varepsilon(0, x) \phi^\varepsilon(0, x) dx \\
& + \frac{1}{\varepsilon} \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} v_\alpha^\varepsilon \left(\varphi_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon} \rho_\alpha^\varepsilon b^* - \tilde{b}_\alpha^\varepsilon \right) \cdot \nabla_x \phi^\varepsilon dx dt \\
& - \frac{1}{\varepsilon} \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \operatorname{div}(\tilde{b}_\alpha^\varepsilon) v_\alpha^\varepsilon \phi^\varepsilon dx dt + \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \tilde{D}_\alpha^\varepsilon \nabla v_\alpha^\varepsilon \cdot \nabla_x \phi^\varepsilon dx dt \\
& + \frac{1}{\varepsilon^2} \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_0^T \int_{\mathbb{R}^d} \Pi_{\alpha\beta}^\varepsilon \varphi_\alpha^{*\varepsilon} \varphi_\beta^\varepsilon (v_\beta^\varepsilon - v_\alpha^\varepsilon) \phi^\varepsilon dx dt + \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \left(\tilde{b}_\alpha^\varepsilon \cdot \nabla v_\alpha^\varepsilon \right) \phi_{1,\alpha}^\varepsilon dx dt \\
& + \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \varphi_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon} \rho_\alpha^\varepsilon v_\alpha^\varepsilon b^* \cdot \nabla \phi_{1,\alpha}^\varepsilon dx dt + \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \tilde{D}_\alpha^\varepsilon \nabla v_\alpha^\varepsilon \cdot \nabla_y \phi_{1,\alpha}^\varepsilon dx dt \\
& + \frac{1}{\varepsilon} \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_0^T \int_{\mathbb{R}^d} \Pi_{\alpha\beta}^\varepsilon \varphi_\alpha^{*\varepsilon} \varphi_\beta^\varepsilon (v_\beta^\varepsilon - v_\alpha^\varepsilon) \phi_{1,\alpha}^\varepsilon dx dt + \mathcal{O}(\varepsilon) = 0.
\end{aligned}$$

In a first step we choose $\phi^\varepsilon \equiv 0$ in (6.10) and pass to the limit as $\varepsilon \rightarrow 0$ which yields:

$$\begin{aligned}
(6.11) \quad & - \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \int_Y \varphi_\alpha \varphi_\alpha^* \rho_\alpha b^* \cdot \nabla_x v \phi_{1,\alpha} dy dx dt \\
& + \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \int_Y \tilde{b}_\alpha \cdot \left(\nabla_x v + \nabla_y v_{1,\alpha} \right) \phi_{1,\alpha} dy dx dt \\
& - \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \int_Y \operatorname{div}_y \left(\tilde{D}_\alpha \left(\nabla_x v + \nabla_y v_{1,\alpha} \right) \right) \phi_{1,\alpha} dy dx dt \\
& + \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_0^T \int_{\mathbb{R}^d} \int_Y \Pi_{\alpha\beta} \varphi_\alpha^* \varphi_\beta (v_{1,\beta} - v_{1,\alpha}) \phi_{1,\alpha}(t, x, y) dy dx dt = 0.
\end{aligned}$$

The above expression is the variational formulation for the following PDE:

$$(6.12) \quad \begin{cases} \tilde{b}_\alpha \cdot \left(\nabla_y v_{1,\alpha} + \nabla_x v \right) - \operatorname{div}_y \left(\tilde{D}_\alpha \left(\nabla_y v_{1,\alpha} + \nabla_x v \right) \right) \\ + \sum_{\beta=1}^N \Pi_{\alpha\beta} \varphi_\alpha^* \varphi_\beta (v_{1,\beta} - v_{1,\alpha}) = \varphi_\alpha \varphi_\alpha^* \rho_\alpha b^* \cdot \nabla_x v & \text{in } Y, \\ y \rightarrow v_{1,\alpha}(y) & Y\text{-periodic,} \end{cases}$$

for every $1 \leq \alpha \leq N$. By the Fredholm result of Lemma 16, we have the existence and uniqueness of $(v_{1,\alpha})_{1 \leq \alpha \leq N} \in L^2((0, T) \times \mathbb{R}^d; \mathcal{H}(Y))$ if and only if the compatibility condition (6.2) is satisfied. Writing down the compatibility

condition for (6.12) yields the expression (6.7) for the drift velocity b^* . Also by linearity of (6.12), we deduce that we can separate the slow and fast variables in $v_{1,\alpha}$ as in (6.5) with $(\omega_{i,\alpha})_{1 \leq \alpha \leq N}$ satisfying the coupled cell problem (6.6).

In a second step we choose $\phi_{1,\alpha}^\varepsilon \equiv 0$ in (6.10) and substitute (4.8) for the initial data $v_\alpha^\varepsilon(0, x)$, which yields

$$\begin{aligned}
 (6.13) \quad & - \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \varphi_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon} \rho_\alpha^\varepsilon v_\alpha^\varepsilon \frac{\partial \phi^\varepsilon}{\partial t} dx dt - \sum_{\alpha=1}^N \int_{\mathbb{R}^d} \varphi_\alpha^{*\varepsilon} \rho_\alpha^\varepsilon u_\alpha^{in}(x) \phi^\varepsilon(0, x) dx \\
 & + \frac{1}{\varepsilon} \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \varphi_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon} v_\alpha^\varepsilon \rho_\alpha^\varepsilon b^* \cdot \nabla_x \phi^\varepsilon dx dt - \frac{1}{\varepsilon} \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} v_\alpha^\varepsilon \tilde{b}_\alpha^\varepsilon \cdot \nabla_x \phi^\varepsilon dx dt \\
 & - \frac{1}{\varepsilon} \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \operatorname{div}(\tilde{b}_\alpha^\varepsilon) v_\alpha^\varepsilon \phi^\varepsilon dx dt + \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \tilde{D}_\alpha^\varepsilon \nabla v_\alpha^\varepsilon \cdot \nabla_x \phi^\varepsilon dx dt \\
 & + \frac{1}{\varepsilon^2} \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_0^T \int_{\mathbb{R}^d} \Pi_{\alpha\beta}^\varepsilon \varphi_\alpha^{*\varepsilon} \varphi_\beta^\varepsilon (v_\beta^\varepsilon - v_\alpha^\varepsilon) \phi^\varepsilon dx dt = 0.
 \end{aligned}$$

Using the expression (4.11) for the divergence of \tilde{b}_α allows us to obtain

$$\begin{aligned}
 (6.14) \quad & - \frac{1}{\varepsilon} \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \operatorname{div}(\tilde{b}_\alpha^\varepsilon) v_\alpha^\varepsilon \phi^\varepsilon dx dt + \frac{1}{\varepsilon^2} \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_0^T \int_{\mathbb{R}^d} \Pi_{\alpha\beta}^\varepsilon \varphi_\alpha^{*\varepsilon} \varphi_\beta^\varepsilon (v_\beta^\varepsilon - v_\alpha^\varepsilon) \phi^\varepsilon dx dt \\
 & = \frac{1}{\varepsilon^2} \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_0^T \int_{\mathbb{R}^d} \left(\Pi_{\alpha\beta}^\varepsilon \varphi_\alpha^{*\varepsilon} \varphi_\beta^\varepsilon v_\alpha^\varepsilon - \Pi_{\alpha\beta}^{*\varepsilon} \varphi_\alpha^\varepsilon \varphi_\beta^{*\varepsilon} v_\alpha^\varepsilon \right) \phi^\varepsilon dx dt \\
 & + \frac{1}{\varepsilon^2} \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_0^T \int_{\mathbb{R}^d} \left(\Pi_{\alpha\beta}^\varepsilon \varphi_\alpha^{*\varepsilon} \varphi_\beta^\varepsilon v_\beta^\varepsilon - \Pi_{\alpha\beta}^\varepsilon \varphi_\alpha^{*\varepsilon} \varphi_\beta^{*\varepsilon} v_\alpha^\varepsilon \right) \phi^\varepsilon dx dt = 0.
 \end{aligned}$$

Thanks to (6.14) all terms of order $\mathcal{O}(\varepsilon^{-2})$ in (6.13) cancel each other. There are, however, terms of $\mathcal{O}(\varepsilon^{-1})$ in (6.13) which still prevent us to pass to the limit as $\varepsilon \rightarrow 0$. In order to remedy the situation, we introduce the following auxiliary problem posed in the unit cell:

$$(6.15) \quad \begin{cases} -\Delta \Xi = \sum_{\alpha=1}^N \left(\varphi_\alpha \varphi_\alpha^* \rho_\alpha b^* - \tilde{b}_\alpha \right) & \text{in } Y, \\ y \rightarrow \Xi(y) & Y\text{-periodic.} \end{cases}$$

The above auxiliary problem is well-posed, thanks to our choice (6.7) of the drift velocity and the chosen normalization (4.3). We scale (6.15) to the entire domain via the change of variables $y \rightarrow \varepsilon^{-1}x$. The vector-valued function $\Xi^\varepsilon(x) = \Xi(x/\varepsilon)$

satisfies

$$(6.16) \quad \begin{cases} -\varepsilon^2 \Delta \Xi^\varepsilon = \sum_{\alpha=1}^N \left(\varphi_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon} \rho_\alpha^\varepsilon b^* - \tilde{b}_\alpha^\varepsilon \right) & \text{in } \mathbb{R}^d, \\ x \rightarrow \Xi^\varepsilon & \varepsilon Y\text{-periodic.} \end{cases}$$

Getting back to the variational formulation (6.13), let us regroup the problematic terms of order $\mathcal{O}(\varepsilon^{-1})$:

$$(6.17) \quad \begin{aligned} & \frac{1}{\varepsilon} \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} v_\alpha^\varepsilon \left(\varphi_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon} v_\alpha^\varepsilon \rho_\alpha^\varepsilon b^* - b_\alpha^\varepsilon \right) \cdot \nabla_x \phi^\varepsilon dx dt \\ &= -\frac{\varepsilon^2}{\varepsilon} \sum_{i=1}^d \int_0^T \int_{\mathbb{R}^d} \Delta \Xi_i^\varepsilon \frac{\partial \phi^\varepsilon}{\partial x_i} v_\alpha^\varepsilon dx dt + \frac{1}{\varepsilon} \sum_{\beta=1}^N \int_0^T \int_{\mathbb{R}^d} (v_\alpha^\varepsilon - v_\beta^\varepsilon) \tilde{b}_\beta^\varepsilon \cdot \nabla_x \phi^\varepsilon dx dt \\ &+ \frac{1}{\varepsilon} \sum_{\beta=1}^N \int_0^T \int_{\mathbb{R}^d} \varphi_\beta^\varepsilon \varphi_\beta^{*\varepsilon} \rho_\beta^\varepsilon (v_\beta^\varepsilon - v_\alpha^\varepsilon) b^* \cdot \nabla_x \phi^\varepsilon dx dt \\ (6.18) \quad &= \varepsilon \sum_{i=1}^d \int_0^T \int_{\mathbb{R}^d} \nabla \Xi_i^\varepsilon \cdot \nabla \left(\frac{\partial \phi^\varepsilon}{\partial x_i} \right) v_\alpha^\varepsilon dx dt + \varepsilon \sum_{i=1}^d \int_0^T \int_{\mathbb{R}^d} \nabla \Xi_i^\varepsilon \cdot \nabla v_\alpha^\varepsilon \left(\frac{\partial \phi^\varepsilon}{\partial x_i} \right) dx dt \\ &+ \frac{1}{\varepsilon} \sum_{\beta=1}^N \int_0^T \int_{\mathbb{R}^d} \varphi_\beta^\varepsilon \varphi_\beta^{*\varepsilon} \rho_\beta^\varepsilon (v_\beta^\varepsilon - v_\alpha^\varepsilon) b^* \cdot \nabla_x \phi^\varepsilon dx dt \\ &+ \frac{1}{\varepsilon} \sum_{\beta=1}^N \int_0^T \int_{\mathbb{R}^d} (v_\alpha^\varepsilon - v_\beta^\varepsilon) \tilde{b}_\beta^\varepsilon \cdot \nabla_x \phi^\varepsilon dx dt, \end{aligned}$$

where we have used the scaled auxiliary problem (6.16). We can now pass to the limit in (6.18) since the sequences $(v_\beta^\varepsilon - v_\alpha^\varepsilon)/\varepsilon$ are bounded. Taking into consideration (6.14) and (6.18), the variational formulation (6.13) rewrites as

$$(6.19) \quad \begin{aligned} & - \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \varphi_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon} \rho_\alpha^\varepsilon v_\alpha^\varepsilon \frac{\partial \phi^\varepsilon}{\partial t} dx dt - \sum_{\alpha=1}^N \int_{\mathbb{R}^d} \varphi_\alpha^{*\varepsilon} \rho_\alpha^\varepsilon u_\alpha^{in}(x) \phi^\varepsilon(0, x) dx \\ & + \varepsilon \sum_{i=1}^d \int_0^T \int_{\mathbb{R}^d} \nabla \Xi_i^\varepsilon \cdot \nabla \left(\frac{\partial \phi^\varepsilon}{\partial x_i} \right) v_\alpha^\varepsilon dx dt + \varepsilon \sum_{i=1}^d \int_0^T \int_{\mathbb{R}^d} \nabla \Xi_i^\varepsilon \cdot \nabla v_\alpha^\varepsilon \left(\frac{\partial \phi^\varepsilon}{\partial x_i} \right) dx dt \\ & + \frac{1}{\varepsilon} \sum_{\beta=1}^N \int_0^T \int_{\mathbb{R}^d} \varphi_\beta^\varepsilon \varphi_\beta^{*\varepsilon} \rho_\beta^\varepsilon (v_\beta^\varepsilon - v_\alpha^\varepsilon) b^* \cdot \nabla_x \phi^\varepsilon dx dt + \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \tilde{D}_\alpha^\varepsilon \nabla v_\alpha^\varepsilon \cdot \nabla_x \phi^\varepsilon dx dt \\ & + \frac{1}{\varepsilon} \sum_{\beta=1}^N \int_0^T \int_{\mathbb{R}^d} (v_\alpha^\varepsilon - v_\beta^\varepsilon) \tilde{b}_\beta^\varepsilon \cdot \nabla_x \phi^\varepsilon dx dt = 0. \end{aligned}$$

Using the compactness results from Theorem 15, we pass to the limit as $\varepsilon \rightarrow 0$ in the above variational formulation leading to:

$$\begin{aligned}
(6.20) \quad & - \int_0^T \int_{\mathbb{R}^d} v \frac{\partial \phi}{\partial t} dx dt - \sum_{\alpha=1}^N \int_{\mathbb{R}^d} \int_Y u_\alpha^{in}(x) \phi(0, x) \varphi_\alpha^* \rho_\alpha dy dx \\
& + \sum_{\alpha=1}^N \int_0^T \int_{\mathbb{R}^d} \int_Y \tilde{D}_\alpha \left(\nabla v + \nabla_y v_{1,\alpha} \right) \cdot \nabla_x \phi dy dx dt \\
& + \sum_{i=1}^d \int_0^T \int_{\mathbb{R}^d} \int_Y \nabla_y \Xi_i \cdot \nabla_y v_{1,\alpha} \frac{\partial \phi}{\partial x_i} dy dx dt \\
& + \sum_{\beta=1}^N \int_0^T \int_{\mathbb{R}^d} \int_Y \varphi_\beta \varphi_\beta^* \rho_\beta \left(v_{1,\beta} - v_{1,\alpha} \right) b^* \cdot \nabla_x \phi dy dx dt \\
& + \sum_{\beta=1}^N \int_0^T \int_{\mathbb{R}^d} \int_Y \left(v_{1,\alpha} - v_{1,\beta} \right) \tilde{b}_\beta \cdot \nabla_x \phi dy dx dt = 0.
\end{aligned}$$

Substituting (6.5) for $v_{1,\alpha}$ in (6.20), we obtain

$$\begin{aligned}
(6.21) \quad & - \int_0^T \int_{\mathbb{R}^d} v \frac{\partial \phi}{\partial t} dx dt - \sum_{\alpha=1}^N \int_{\mathbb{R}^d} u_\alpha^{in}(x) \phi(0, x) \int_Y \varphi_\alpha^*(y) \rho_\alpha(y) dy dx \\
& + \sum_{\alpha=1}^N \sum_{i,j=1}^d \int_0^T \int_{\mathbb{R}^d} \frac{\partial v}{\partial x_j} \frac{\partial \phi}{\partial x_i} \int_Y \tilde{D}_\alpha \left(\nabla y_j + \nabla_y \omega_{j,\alpha} \right) \cdot \nabla y_i dy dx dt \\
& - \sum_{i,j=1}^d \int_0^T \int_{\mathbb{R}^d} \frac{\partial v}{\partial x_j} \frac{\partial \phi}{\partial x_i} \int_Y \left(\Delta_y \Xi_i \right) \omega_{j,\alpha} dy dx dt \\
& + \sum_{\beta=1}^N \sum_{i,j=1}^d \int_0^T \int_{\mathbb{R}^d} \frac{\partial v}{\partial x_j} \frac{\partial \phi}{\partial x_i} \int_Y \varphi_\beta \varphi_\beta^* \rho_\beta \left(\omega_{j,\beta} - \omega_{j,\alpha} \right) b^* \cdot \nabla y_i dy dx dt \\
& + \sum_{\beta=1}^N \sum_{i,j=1}^d \int_0^T \int_{\mathbb{R}^d} \frac{\partial v}{\partial x_j} \frac{\partial \phi}{\partial x_i} \int_Y \left(\omega_{j,\alpha} - \omega_{j,\beta} \right) \tilde{b}_\beta \cdot \nabla y_i dy dx dt = 0.
\end{aligned}$$

Using the information from the auxiliary cell problem (6.15) in (6.21) and making a rearrangement similar to that of (6.17), we deduce that (6.21) is nothing but the variational formulation for a scalar diffusion equation (6.8) for $v(t, x)$ with the entries of the diffusion matrix given by

$$\mathcal{D}_{ij} = \sum_{\alpha=1}^N \int_Y \tilde{D}_\alpha \left(\nabla y_j + \nabla_y \omega_{j,\alpha} \right) \cdot \nabla y_i dy + \sum_{\alpha=1}^N \int_Y \omega_{j,\alpha} \left(\varphi_\alpha \varphi_\alpha^* v_\alpha^\varepsilon \rho_\alpha^\varepsilon b^* - b_\alpha^\varepsilon \right) \cdot e_i dy.$$

By integration by parts, it is clear that the diffusion matrix \mathcal{D} is contracted with the Hessian matrix $\nabla \nabla v$, which is symmetric. Thus the non-symmetric part of \mathcal{D} does not contribute to the homogenized equation (6.8). So, the above expression

for the diffusion matrix is symmetrized:

(6.22)

$$\begin{aligned} \mathcal{D}_{ij} = & \sum_{\alpha=1}^N \int_Y \tilde{D}_\alpha e_j \cdot e_i dy + \frac{1}{2} \left\{ \sum_{\alpha=1}^N \int_Y \left(\tilde{D}_\alpha \nabla_y \omega_{i,\alpha} \cdot e_j + \tilde{D}_\alpha \nabla_y \omega_{j,\alpha} \cdot e_i \right) dy \right\} \\ & + \frac{1}{2} \left\{ \sum_{\alpha=1}^N \int_Y \left(\omega_{i,\alpha} (\varphi_\alpha \varphi_\alpha^* \rho_\alpha b^* - \tilde{b}_\alpha) \cdot e_j + \omega_{j,\alpha} (\varphi_\alpha \varphi_\alpha^* \rho_\alpha b^* - \tilde{b}_\alpha) \cdot e_i \right) dy \right\}. \end{aligned}$$

To obtain the desired expression (6.9) for the diffusion matrix, we consider the variational formulation for the cell problem (6.6) with test functions $(\psi_\alpha)_{1 \leq \alpha \leq N}$

(6.23)

$$\begin{aligned} & \sum_{\alpha=1}^N \int_Y \left(\tilde{b}_\alpha \cdot \nabla_y \omega_{i,\alpha} \right) \psi_\alpha dy + \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_Y \Pi_{\alpha\beta} \varphi_\alpha^* \varphi_\beta \left(\omega_{i,\beta} - \omega_{i,\alpha} \right) \psi_\alpha dy \\ & + \sum_{\alpha=1}^N \int_Y \tilde{D}_\alpha \left(\nabla_y \omega_{i,\alpha} + e_i \right) \cdot \nabla_y \psi_\alpha dy = \sum_{\alpha=1}^N \int_Y \left(\varphi_\alpha \varphi_\alpha^* \rho_\alpha b^* - \tilde{b}_\alpha \right) \cdot e_i \psi_\alpha dy. \end{aligned}$$

In (6.23) we first choose the test function $(\psi_\alpha) = (\omega_{j,\alpha})$. Similarly, in (6.23) for j instead of i , we choose the test function $(\psi_\alpha) = (\omega_{i,\alpha})$. This leads to

(6.24)

$$\begin{aligned} & \frac{1}{2} \left\{ \sum_{\alpha=1}^N \int_Y \left(\omega_{i,\alpha} (\varphi_\alpha \varphi_\alpha^* \rho_\alpha b^* - \tilde{b}_\alpha) \cdot e_j + \omega_{j,\alpha} (\varphi_\alpha \varphi_\alpha^* \rho_\alpha b^* - \tilde{b}_\alpha) \cdot e_i \right) dy \right\} \\ & = \sum_{\alpha=1}^N \int_Y \tilde{D}_\alpha \nabla_y \omega_{i,\alpha} \cdot \nabla_y \omega_{j,\alpha} dy + \frac{1}{2} \left\{ \sum_{\alpha=1}^N \int_Y \left(\tilde{D}_\alpha \nabla_y \omega_{i,\alpha} \cdot e_j + \tilde{D}_\alpha \nabla_y \omega_{j,\alpha} \cdot e_i \right) dy \right\} \\ & - \frac{1}{2} \left\{ \sum_{\alpha=1}^N \int_Y \omega_{i,\alpha} \omega_{j,\alpha} \operatorname{div}_y \tilde{b}_\alpha dy \right\} \\ & + \frac{1}{2} \left\{ \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_Y \left(\Pi_{\alpha\beta} \varphi_\alpha^* \varphi_\beta (\omega_{i,\beta} - \omega_{i,\alpha}) \omega_{j,\alpha} + \Pi_{\alpha\beta} \varphi_\alpha^* \varphi_\beta (\omega_{j,\beta} - \omega_{j,\alpha}) \omega_{i,\alpha} \right) dy \right\}. \end{aligned}$$

Using formula (4.11) for the divergence of \tilde{b}_α in (6.24), its right hand side simplifies as

(6.25)

$$\begin{aligned} & \sum_{\alpha=1}^N \int_Y \tilde{D}_\alpha \nabla_y \omega_{i,\alpha} \cdot \nabla_y \omega_{j,\alpha} dy + \frac{1}{2} \left\{ \sum_{\alpha=1}^N \int_Y \left(\tilde{D}_\alpha \nabla_y \omega_{i,\alpha} \cdot e_j + \tilde{D}_\alpha \nabla_y \omega_{j,\alpha} \cdot e_i \right) dy \right\} \\ & - \frac{1}{2} \sum_{\alpha=1}^N \sum_{\beta=1}^N \int_Y \varphi_\alpha^* \varphi_\beta \Pi_{\alpha\beta} \left(\omega_{i,\alpha} - \omega_{i,\beta} \right) \left(\omega_{j,\alpha} - \omega_{j,\beta} \right) dy. \end{aligned}$$

Plugging (6.25) in the symmetrized formula (6.22) leads to the desired equation (6.9). Eventually the scalar homogenized equation (6.8) has a unique solution since, by virtue of Lemma 24, the dispersion matrix is positive definite. This

guarantees that the entire sequence v_α^ε converges to v , for $1 \leq \alpha \leq N$, and not merely a subsequence as in Theorem 15. \square

7. ADSORPTION IN POROUS MEDIA

In this section, we give a generalization of our previous result in a more applied context. Our goal is to upscale a model of multicomponent transport in an highly heterogeneous porous medium in presence of adsorption reaction at the fluid-pore interface. In [7], the authors study the homogenization of one single scalar convection-diffusion-reaction equation posed in an ε -periodic infinite porous medium:

$$(7.1) \quad \begin{cases} \rho^\varepsilon \frac{\partial u^\varepsilon}{\partial t} + \frac{1}{\varepsilon} b^\varepsilon \cdot \nabla u^\varepsilon - \operatorname{div}(D^\varepsilon \nabla u^\varepsilon) + \frac{1}{\varepsilon^2} c^\varepsilon u^\varepsilon = 0 & \text{in } (0, T) \times \Omega_\varepsilon, \\ -D^\varepsilon \nabla u^\varepsilon \cdot n = \frac{1}{\varepsilon} \kappa u^\varepsilon & \text{on } (0, T) \times \partial\Omega_\varepsilon. \end{cases}$$

Typically, an ε -periodic infinite porous medium is built out of \mathbb{R}^d ($d = 2$ or 3 , being the space dimension) by removing a periodic distribution of solid obstacles which, after rescaling, are all similar to the unit obstacle Σ^0 . More precisely, let $Y =]0, 1[^d$ be the unit periodicity cell. Let us consider a smooth partition $Y = \Sigma^0 \cup Y^0$ where Σ^0 is the solid part and Y^0 is the fluid part. The fluid part (extended by periodicity) is assumed to be a smooth connected open subset whereas no particular assumptions are made on the solid part. For each multi-index $j \in \mathbb{Z}^d$, we define $Y_\varepsilon^j := \varepsilon(Y^0 + j)$, $\Sigma_\varepsilon^j := \varepsilon(\Sigma^0 + j)$, $S_\varepsilon^j := \varepsilon(\partial\Sigma^0 + j)$, the periodic porous medium $\Omega_\varepsilon := \cup_{j \in \mathbb{Z}^d} Y_\varepsilon^j$ and the $(d-1)$ -dimensional surface $\partial\Omega_\varepsilon := \cup_{j \in \mathbb{Z}^d} S_\varepsilon^j$.

In this section, we generalize the results of [7] to the multicomponent case. We consider the following weakly coupled cooperative parabolic system with Neumann boundary condition at the fluid-pore interface.

$$(7.2) \quad \begin{cases} \rho_\alpha^\varepsilon \frac{\partial u_\alpha^\varepsilon}{\partial t} + \frac{1}{\varepsilon} b_\alpha^\varepsilon \cdot \nabla u_\alpha^\varepsilon - \operatorname{div}(D_\alpha^\varepsilon \nabla u_\alpha^\varepsilon) = 0 & \text{in } (0, T) \times \Omega_\varepsilon, \\ -D_\alpha^\varepsilon \nabla u_\alpha^\varepsilon \cdot n = \frac{1}{\varepsilon} \sum_{\beta=1}^N \Pi_{\alpha\beta}^\varepsilon u_\beta^\varepsilon & \text{on } (0, T) \times \partial\Omega_\varepsilon, \\ u_\alpha^\varepsilon(0, x) = u_\alpha^{in}(x) & \text{in } \Omega_\varepsilon. \end{cases}$$

Remark 25. Note the different scaling in front of the surface reaction terms. It is of order ε^{-1} because it balances a flux rather than a diffusive term, as in the previous model of Section 2. As usual, by the change of variable $(\tau, y) \rightarrow (\varepsilon^{-2}t, \varepsilon^{-1}x)$ all singular powers of ε disappears in (7.2) written in the (τ, y) variables.

The hypotheses on the coefficients in (7.2) are exactly the same as in Section 2. As before it is impossible to obtain uniform (in ε) estimates on the solutions

u_α^ε of (7.2). As was done in Section 4, we employ the method of factorization by introducing a new unknown:

$$v_\alpha^\varepsilon(t, x) = \exp(\lambda t / \varepsilon^2) \frac{u_\alpha^\varepsilon(t, x)}{\varphi_\alpha\left(\frac{x}{\varepsilon}\right)},$$

where $(\lambda, \varphi_\alpha)$ and $(\lambda, \varphi_\alpha^*)$ are the principal eigenpairs associated with the (new) following spectral problems respectively:

$$(7.3) \quad \begin{cases} b_\alpha(y) \cdot \nabla_y \varphi_\alpha - \operatorname{div}_y (D_\alpha \nabla_y \varphi_\alpha) = \lambda \rho_\alpha \varphi_\alpha & \text{in } Y^0, \\ -D_\alpha \nabla_y \varphi_\alpha \cdot n = \sum_{\beta=1}^N \Pi_{\alpha\beta} \varphi_\beta & \text{on } \partial\Sigma^0, \\ y \rightarrow \varphi_\alpha(y) & Y\text{-periodic.} \end{cases}$$

$$(7.4) \quad \begin{cases} -\operatorname{div}_y (b_\alpha \varphi_\alpha^*) - \operatorname{div}_y (D_\alpha \nabla_y \varphi_\alpha^*) = \lambda \rho_\alpha \varphi_\alpha^* & \text{in } Y^0, \\ -D_\alpha \nabla_y \varphi_\alpha^* \cdot n - b_\alpha(y) \cdot n \varphi_\alpha^* = \sum_{\beta=1}^N \Pi_{\alpha\beta}^* \varphi_\beta^* & \text{on } \partial\Sigma^0, \\ y \rightarrow \varphi_\alpha^*(y) & Y\text{-periodic.} \end{cases}$$

Proposition 5, which guarantees the existence of principal eigenpairs for the spectral problems (4.1)-(4.2), carries over to the above spectral problems (7.3)-(7.4) as well. This is apparent from the proofs in [24, 22]. The normalization (ensuring uniqueness of the eigenfunctions) that we choose is:

$$\sum_{\alpha=1}^N \int_{Y^0} \varphi_\alpha \varphi_\alpha^* \rho_\alpha \, dy = 1.$$

As in Section 4 it is a matter of simple algebra to obtain the factorized system for (7.2) with the new unknown which is, for each $1 \leq \alpha \leq N$,

$$(7.5) \quad \begin{cases} \varphi_\alpha^\varepsilon \varphi_\alpha^{*\varepsilon} \rho_\alpha^\varepsilon \frac{\partial v_\alpha^\varepsilon}{\partial t} + \frac{1}{\varepsilon} \tilde{b}_\alpha^\varepsilon \cdot \nabla v_\alpha^\varepsilon - \operatorname{div} \left(\tilde{D}_\alpha^\varepsilon \nabla v_\alpha^\varepsilon \right) = 0 & \text{in } (0, T) \times \Omega_\varepsilon, \\ -\tilde{D}_\alpha^\varepsilon \nabla v_\alpha^\varepsilon \cdot n = \frac{1}{\varepsilon} \sum_{\beta=1}^N \Pi_{\alpha\beta}^\varepsilon \varphi_\alpha^{*\varepsilon} \varphi_\beta^\varepsilon (v_\beta^\varepsilon - v_\alpha^\varepsilon) & \text{on } (0, T) \times \partial\Omega_\varepsilon, \\ v_\alpha^\varepsilon(0, x) = \frac{u_\alpha^{\text{in}}(x)}{\varphi_\alpha\left(\frac{x}{\varepsilon}\right)} & \text{in } \Omega_\varepsilon, \end{cases}$$

where the convective fields \tilde{b}_α and diffusion matrices \tilde{D}_α are given by the same formulae (4.9) and (4.10). A proof, completely similar to that of Lemma 10,

yields the following a priori estimates (7.5):

$$(7.6) \quad \begin{aligned} & \sum_{\alpha=1}^N \|v_\alpha^\varepsilon\|_{L^\infty((0,T);L^2(\Omega_\varepsilon))} + \sum_{\alpha=1}^N \|\nabla v_\alpha^\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} \\ & + \sum_{\alpha=1}^N \sum_{\beta=1}^N \sqrt{\varepsilon} \left\| \frac{1}{\varepsilon} (v_\alpha^\varepsilon - v_\beta^\varepsilon) \right\|_{L^2((0,T)\times\partial\Omega_\varepsilon)} \leq C \sum_{\alpha=1}^N \|v_\alpha^{in}\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Remark 26. Since the $(d-1)$ dimensional measure of the periodic surface $\partial\Omega_\varepsilon$ is of order $\mathcal{O}(\varepsilon^{-1})$, a bound of the type $\sqrt{\varepsilon}\|z_\varepsilon\|_{L^2(\partial\Omega_\varepsilon)} \leq C$ means that the sequence z_ε is bounded on the surface $\partial\Omega_\varepsilon$.

In the a priori estimates (7.6), we have bounds in function spaces defined on the periodic surface $\partial\Omega_\varepsilon$. In order to speak of the convergence of sequences in such function spaces, we need to generalize the Definition 11 of two-scale convergence with drift for periodic surfaces. This generalization was introduced in [16]. We state this definition together with the corresponding compactness result (the proof of which is similar to that of Theorem 9.1 in [2]).

Lemma 27. Let $b^* \in \mathbb{R}^d$ be a constant vector. Suppose that $u_\varepsilon(t, x)$ is a sequence of functions uniformly bounded in $L^2((0, T) \times \partial\Omega_\varepsilon)$ in the sense that

$$\sqrt{\varepsilon}\|u_\varepsilon\|_{L^2((0,T)\times\partial\Omega_\varepsilon)} \leq C.$$

Then, there exists a subsequence, still denoted by $u_\varepsilon(t, x)$, and a function $u_0(t, x, y) \in L^2((0, T) \times \mathbb{R}^d \times \partial\Sigma^0)$ such that

$$(7.7) \quad \begin{aligned} & \lim_{\varepsilon \rightarrow 0} \varepsilon \int_0^T \int_{\partial\Omega_\varepsilon} u_\varepsilon(t, x) \phi\left(t, x - \frac{b^*}{\varepsilon}t, \frac{x}{\varepsilon}\right) d\sigma_\varepsilon(x) dt \\ & = \int_0^T \int_{\mathbb{R}^d} \int_{\partial\Sigma^0} u_0(t, x, y) \phi(t, x, y) d\sigma(y) dx dt, \end{aligned}$$

for any function $\phi(t, x, y) \in C_c^\infty((0, T) \times \mathbb{R}^d; C_\#^\infty(Y))$.

In (7.7), $d\sigma_\varepsilon(x)$ and $d\sigma(y)$ denote the standard surface measures on $\partial\Omega_\varepsilon$ and $\partial\Sigma^0$ respectively. We denote this convergence on periodic surfaces in moving coordinates by $u_\varepsilon \xrightarrow{2s-\text{drift}} u_0$.

Remark 28. Let $u_\varepsilon(t, x)$ be a sequence of functions defined on $(0, T) \times \Omega_\varepsilon$. Let γ be the trace operator, i.e., $\gamma u = u|_{\partial\Omega_\varepsilon}$. Suppose that we have a well-defined sequence of associated trace functions $\gamma u_\varepsilon(t, x)$ on $(0, T) \times \partial\Omega_\varepsilon$. If $u_\varepsilon \xrightarrow{2-\text{drift}} u_0$ and $\gamma u_\varepsilon \xrightarrow{2s-\text{drift}} v_0$ with the same drift velocity for both convergences, then $\gamma u_0 = v_0$ i.e., $\gamma u_0 = u_0|_{\partial\Sigma^0} = v_0$ (see [5] for details). In the sequel we systematically identify the “bulk” and “surface” two-scale limits.

We now define the homogenized velocity which is chosen as the constant drift in the definition of two-scale convergence with drift:

$$(7.8) \quad b^* = \sum_{\alpha=1}^N \int_{Y^0} \tilde{b}_\alpha(y) dy.$$

Theorem 29. *Let $(v_\alpha^\varepsilon)_{1 \leq \alpha \leq N} \in L^2((0, T); H^1(\Omega_\varepsilon))^N$ be the sequence of solutions of (7.5). Let $b^* \in \mathbb{R}^d$ be given by (7.8). There exist $v \in L^2((0, T); H^1(\mathbb{R}^d))$ and $\omega_{i,\alpha} \in H_{\#}^1(Y^0)$, for $1 \leq \alpha \leq N$ and $1 \leq i \leq d$, such that v_α^ε two-scale converges with drift b^* , as $\varepsilon \rightarrow 0$, in the following sense:*

$$(7.9) \quad \begin{cases} v_\alpha^\varepsilon \xrightarrow{2\text{-drift}} v, \\ \nabla v_\alpha^\varepsilon \xrightarrow{2\text{-drift}} \nabla_x v + \nabla_y \left(\sum_{i=1}^d \omega_{i,\alpha} \frac{\partial v}{\partial x_i} \right), \\ \frac{1}{\varepsilon} (v_\alpha^\varepsilon - v_\beta^\varepsilon) \xrightarrow{2s\text{-drift}} \sum_{i=1}^d (\omega_{i,\alpha} - \omega_{i,\beta}) \frac{\partial v}{\partial x_i}, \end{cases}$$

for every $1 \leq \alpha, \beta \leq N$. The two-scale limit $v(t, x)$ in (7.9) satisfies the following homogenized equation:

$$(7.10) \quad \begin{cases} \frac{\partial v}{\partial t} - \operatorname{div}(\mathcal{D} \nabla v) = 0 & \text{in } (0, T) \times \mathbb{R}^d, \\ v(0, x) = \sum_{\alpha=1}^N u_\alpha^{\text{in}}(x) \int_{Y^0} \rho_\alpha(y) \varphi_\alpha^*(y) dy & \text{in } \mathbb{R}^d, \end{cases}$$

where the dispersion tensor \mathcal{D} is given by

$$(7.11) \quad \begin{aligned} \mathcal{D}_{ij} &= \sum_{\alpha=1}^N \int_{Y^0} \tilde{D}_\alpha (\nabla_y \omega_{i,\alpha} + e_i) \cdot (\nabla_y \omega_{j,\alpha} + e_j) dy \\ &\quad - \frac{1}{2} \sum_{\alpha,\beta=1}^N \int_{\partial \Sigma^0} \varphi_\alpha^* \varphi_\beta \Pi_{\alpha\beta} (\omega_{i,\alpha} - \omega_{i,\beta}) (\omega_{j,\alpha} - \omega_{j,\beta}) d\sigma(y) \end{aligned}$$

and the components $(\omega_{i,\alpha})_{1 \leq \alpha \leq N}$, for every $1 \leq i \leq d$, are the solutions of the cell problems

$$(7.12) \quad \begin{cases} \tilde{b}_\alpha(y) \cdot (\nabla_y \omega_{i,\alpha} + e_i) - \operatorname{div}_y (\tilde{D}_\alpha (\nabla_y \omega_{i,\alpha} + e_i)) = \varphi_\alpha \varphi_\alpha^* \rho_\alpha b^* \cdot e_i & \text{in } Y^0, \\ -\tilde{D}_\alpha (\nabla_y \omega_{i,\alpha} + e_i) \cdot n = \sum_{\beta=1}^N \Pi_{\alpha\beta} \varphi_\alpha^* \varphi_\beta (\omega_{i,\beta} - \omega_{i,\alpha}) & \text{on } \partial \Sigma^0, \\ y \rightarrow \omega_{i,\alpha} & Y\text{-periodic.} \end{cases}$$

Proof. As we have L^2 bounds on the solution sequence, we have the existence of a subsequence and a two-scale limit, say $(v_\alpha)_{1 \leq \alpha \leq N} \in L^2((0, T); L^2(\mathbb{R}^d))^N$ such that

$$(7.13) \quad v_\alpha^\varepsilon \xrightarrow{2\text{-drift}} v_\alpha$$

for every $1 \leq \alpha \leq N$. For $w \in H^1(\Omega_\varepsilon)$, consider the following Poincaré type inequality derived in [11]:

$$(7.14) \quad \|w\|_{L^2(\Omega_\varepsilon)}^2 \leq C \left(\varepsilon^2 \|\nabla w\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon \|w\|_{L^2(\partial\Omega_\varepsilon)}^2 \right).$$

Taking $w = \frac{1}{\varepsilon} (v_\alpha^\varepsilon - v_\beta^\varepsilon)$, we deduce from (7.14) and a priori estimates (7.6) that

$$(7.15) \quad \sum_{\alpha=1}^d \sum_{\beta=1}^d \|v_\alpha^\varepsilon - v_\beta^\varepsilon\|_{L^2((0, T) \times \Omega_\varepsilon)} \leq C \varepsilon.$$

The above estimate (7.15) implies that the limits obtained in (7.13) do match i.e., $v_\alpha = v$ for every $1 \leq \alpha \leq N$. The H^1 a priori estimate in space as in (7.6) does imply that $v \in L^2((0, T); H^1(\mathbb{R}^d))$ and that there exist limits $v_{1, \alpha} \in L^2((0, T) \times \mathbb{R}^d; H_\#^1(Y^0))$ such that

$$(7.16) \quad \nabla v_\alpha^\varepsilon \xrightarrow{2\text{-drift}} \nabla_x v + \nabla_y v_{1, \alpha}$$

for every $1 \leq \alpha \leq N$. In order to arrive at the two-scale limit of the coupled term on the boundary, we use Lemma 30 below. Taking ϕ from (7.18) as the test function, consider the following expression with the coupled term:

$$\begin{aligned} & \varepsilon \int_0^T \int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon} (v_\alpha^\varepsilon - v_\beta^\varepsilon) \phi \left(t, x - \frac{b^* t}{\varepsilon}, \frac{x}{\varepsilon} \right) d\sigma_\varepsilon(x) dt \\ &= \int_0^T \int_{\Omega_\varepsilon} \operatorname{div} \left((v_\alpha^\varepsilon - v_\beta^\varepsilon) \theta \left(t, x - \frac{b^* t}{\varepsilon}, \frac{x}{\varepsilon} \right) \right) dx dt \\ &= \int_0^T \int_{\Omega_\varepsilon} \left(\nabla v_\alpha^\varepsilon - \nabla v_\beta^\varepsilon \right) \cdot \theta \left(t, x - \frac{b^* t}{\varepsilon}, \frac{x}{\varepsilon} \right) dx dt \\ &+ \int_0^T \int_{\Omega_\varepsilon} \left(v_\alpha^\varepsilon - v_\beta^\varepsilon \right) (\operatorname{div}_x \theta) \left(t, x - \frac{b^* t}{\varepsilon}, \frac{x}{\varepsilon} \right) dx dt \\ &\xrightarrow{2\text{-drift}} \int_0^T \int_{\mathbb{R}^d} \int_{Y^0} \left(\nabla_y v_{1, \alpha} - \nabla_y v_{1, \beta} \right) \cdot \theta dy dx dt \\ &= \int_0^T \int_{\mathbb{R}^d} \int_{\partial\Sigma^0} \left(v_{1, \alpha} - v_{1, \beta} \right) \theta \cdot n d\sigma(y) dx dt = \int_0^T \int_{\mathbb{R}^d} \int_{\partial\Sigma^0} \left(v_{1, \alpha} - v_{1, \beta} \right) \phi d\sigma(y) dx dt, \end{aligned}$$

which implies that

$$\frac{1}{\varepsilon} (v_\alpha^\varepsilon - v_\beta^\varepsilon) \xrightarrow{2s\text{-drift}} v_{1, \alpha} - v_{1, \beta} \quad \text{for every } 1 \leq \alpha, \beta \leq N.$$

The rest of the proof is completely similar to the proof of Theorem 18. We safely leave it to the reader. \square

We finish by stating a technical lemma which was useful in the proof of Theorem 29.

Lemma 30. *For a function $\phi(t, x, y) \in L^2((0, T) \times \mathbb{R}^d \times \partial\Sigma^0)$ such that*

$$(7.17) \quad \int_{\partial\Sigma^0} \phi(t, x, y) d\sigma(y) = 0 \quad \forall (t, x) \in (0, T) \times \mathbb{R}^d,$$

there exists a vector field $\theta(t, x, y) \in L^2((0, T) \times \mathbb{R}^d; L^2_\#(Y^0))^d$ such that

$$(7.18) \quad \begin{cases} \operatorname{div}_y \theta = 0 & \text{in } Y^0, \\ \theta \cdot n = \phi & \text{on } \partial\Sigma^0, \\ y \rightarrow \theta(t, x, y) & Y\text{-periodic.} \end{cases}$$

Proof. Consider the following stationary diffusion problem posed in the unit cell:

$$\begin{aligned} \Delta_y \xi(y) &= 0 & \text{in } Y^0, \\ \nabla_y \xi \cdot n &= \phi & \text{on } \partial\Sigma^0, \end{aligned}$$

with Y -periodic boundary conditions and the Neumann data ϕ satisfying (7.17). The existence and uniqueness of $\xi \in H^1_\#(Y^0)/\mathbb{R}$ is guaranteed for the above problem as (7.17) is indeed the compatibility condition from the Fredholm alternative. Choosing $\theta = \nabla_y \xi$ gives one possible solution for (7.18). \square

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CMAP, UMR CRNS 7641, ÉCOLE POLYTECHNIQUE, ROUTE DE SACLAY, PALAISEAU F91128, FRANCE

E-MAIL: gregoire.allaire@polytechnique.fr

DPMMS, CMS, UNIVERSITY OF CAMBRIDGE, WILBERFORCE ROAD, CAMBRIDGE CB3 0WB, UK

E-MAIL: H.Hutridurga@dpmms.cam.ac.uk